

Non-linear Lie conformal algebras with three generators

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Dedicated to our teacher Victor Kac on the occasion of his 65th birthday

Abstract. We classify certain non-linear Lie conformal algebras with three generators, which can be viewed as deformations of the current Lie conformal algebra of sl_2 . In doing so we discover an interesting 1-parameter family of non-linear Lie conformal algebras R_{-1}^d ($d \in \mathbb{N}$) and the corresponding freely generated vertex algebras V_{-1}^d , which includes for $d = 1$ the affine vertex algebra of sl_2 at the critical level $k = -2$. We construct free-field realizations of the algebras V_{-1}^d extending the Wakimoto realization of \widehat{sl}_2 at the critical level, and we compute their Zhu algebras.

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1. Introduction

1.1. Vertex algebras and Lie conformal algebras

The notion of a *vertex algebra* [B1] provides an axiomatic algebraic description of the operator product expansion of chiral fields in 2-dimensional conformal field theory. Vertex algebras played an important role in the conceptual understanding of the “monstrous moonshine” [CN, FLM, B2, G], and have also proved useful in the representation theory of infinite-dimensional Lie algebras.

The data of a vertex algebra consist of the space of *states* V (an arbitrary vector superspace), the *vacuum vector* $|0\rangle \in V$, the infinitesimal *translation operator* $T \in \text{End } V$, and a collection \mathcal{F} of $\text{End } V$ -valued *quantum fields*, subject to the axioms formulated below (which are “algebraic” consequences of Wightman’s axioms); see [FLM, K, FB, LL, DSK2]. The quantum fields are linear maps from

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V to $V[[z]][z^{-1}]$, where z is a formal variable, and can be viewed as formal series

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \text{End } V,$$

such that $a_{(n)}b = 0$ for $n \gg 0$ (i.e., for n large enough).

The four axioms of a vertex algebra $(V, |0\rangle, T, \mathcal{F} = \{a^\alpha(z)\})$ are:

$$\begin{aligned} (\text{vacuum axiom}) \quad & T|0\rangle = 0, \\ (\text{translation covariance}) \quad & [T, a^\alpha(z)] = \partial_z a^\alpha(z), \\ (\text{locality}) \quad & (z-w)^N [a^\alpha(z), a^\beta(w)] = 0 \text{ for } N \gg 0, \\ (\text{completeness}) \quad & \text{all vectors } a_{(n_1)}^{\alpha_1} \cdots a_{(n_s)}^{\alpha_s} |0\rangle \text{ linearly span } V. \end{aligned}$$

If we enlarge \mathcal{F} to the maximal collection $\bar{\mathcal{F}}$ of quantum fields for which the axioms still hold, then the map

$$\bar{\mathcal{F}} \rightarrow V, \quad a(z) \mapsto a_{(-1)}|0\rangle$$

is bijective (see e.g. [K, DSK2]). We thus get the *state-field correspondence*, defined as the inverse map

$$V \rightarrow \bar{\mathcal{F}}, \quad a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}.$$

Here and further, we use the customary notation $Y(a, z)$ for the quantum field corresponding to the state $a \in V$. The state-field correspondence allows one to introduce bilinear products on V for each $n \in \mathbb{Z}$ by letting

$$a_{(n)}b = \text{Res}_z z^n Y(a, z)b,$$

where the formal *residue* denotes the coefficient of z^{-1} .

The original Borcherds definition [B1] of a vertex algebra was formulated by taking a vector space V with the vacuum vector $|0\rangle$ and bilinear products $a_{(n)}b$ for each $n \in \mathbb{Z}$, satisfying a simple vacuum identity and other more complicated identities, which can be combined into one cubic identity called the *Borcherds identity* (see [K, Eq. (4.8.3)] and also [FLM, FB, LL]). This identity is somewhat similar to the Jacobi identity, and it is as important for the theory of vertex algebras as the latter is for the theory of Lie algebras.

The *Wick product* (= normally ordered product) in a vertex algebra V is defined as $:ab: = a_{(-1)}b$. Some elementary but useful consequences of the axioms are:

$$a_{(n)}|0\rangle = 0, \quad :a|0: = a, \quad a_{(-n-1)}b = \frac{1}{n!} : (T^n a)b :, \quad a, b \in V, \quad n \in \mathbb{Z}_+, \quad (1.1)$$

where \mathbb{Z}_+ denotes the set of non-negative integers. An important special case of the Borcherds identity is the *commutator formula*

$$[Y(a, z), Y(b, w)] = \sum_{j \in \mathbb{Z}_+} Y(a_{(j)}b, w) \partial_w^j \delta(z-w)/j!, \quad (1.2)$$

where

$$\delta(z - w) = \sum_{m \in \mathbb{Z}} z^{-m-1} w^m$$

is the formal delta-function (note that the sum in the right-hand side of (1.2) is finite because $a_{(j)}b = 0$ for $j \gg 0$). Formula (1.2) is conveniently encoded by the λ -bracket [K, DK]

$$[a_\lambda b] = \text{Res}_z e^{z\lambda} Y(a, z)b = \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} a_{(j)}b,$$

which is a polynomial in λ .

In any vertex algebra, the λ -bracket satisfies the axioms of a *Lie conformal (super) algebra* [K] (also known as a “vertex Lie algebra” [P, FB, DLM2]). This is a $\mathbb{C}[T]$ -module R with a \mathbb{C} -linear map $R \otimes R \rightarrow R[\lambda]$, subject to the following axioms:

$$(\text{sesqui-linearity}) \quad [(Ta)_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda(Tb)] = (\lambda + T)[a_\lambda b], \quad (1.3)$$

$$(\text{skew-symmetry}) \quad [a_\lambda b] = -(-1)^{p(a)p(b)} [b_{-\lambda-T}a], \quad (1.4)$$

$$(\text{Jacobi identity}) \quad [[a_\lambda b]_{\lambda+\mu}c] = [a_\lambda[b_\mu c]] - (-1)^{p(a)p(b)} [b_\mu[a_\lambda c]], \quad (1.5)$$

where $p(a) \in \mathbb{Z}/2\mathbb{Z}$ denotes the parity of an element a . Simple Lie conformal superalgebras were classified in [DK, FK]; their representation theory and cohomology theory were developed in [CK, BKV] and other works.

The λ -bracket and Wick product in a vertex algebra are related by the following identities [K, BK]:

(quasi-commutativity)

$$:ab: - (-1)^{p(a)p(b)} :ba: = \int_{-T}^0 d\lambda [a_\lambda b], \quad (1.6)$$

(quasi-associativity)

$$:(:ab:)c: - :a(:bc): = : \left(\int_0^T d\lambda a \right) [b_\lambda c] : + (-1)^{p(a)p(b)} : \left(\int_0^T d\lambda b \right) [a_\lambda c] :, \quad (1.7)$$

(noncommutative Wick formula)

$$[a_\lambda :bc:] = :[a_\lambda b]c: + (-1)^{p(a)p(b)} :b[a_\lambda c]: + \int_0^\lambda d\mu [[a_\lambda b]_\mu c]. \quad (1.8)$$

Conversely, the above formulas, together with the axioms of a Lie conformal algebra for $[a_\lambda b]$ (and the vacuum and translation covariance properties of the Wick product), provide an equivalent definition of the notion of a vertex algebra [BK].

Throughout the paper, we will use the standard convention that a normally ordered product of more than two factors is taken from right to left; for example

$$:abc: = :a(:bc:):, \quad :abcd: = :a(:bcd:): = :a(:b(:cd:):):. \quad (1.9)$$

From (1.6) and (1.7), one obtains the following useful identity [BK]:

$$:abc: - (-1)^{p(a)p(b)} :bac: = : \left(\int_{-T}^0 d\lambda [a_\lambda b] \right) c :. \quad (1.10)$$

We are also going to need the *right Wick formula* [BK]:

$$\begin{aligned} [:ab:_\lambda c] &= :(e^{T\partial_\lambda} a)[b_\lambda c]: + (-1)^{p(a)p(b)} :(e^{T\partial_\lambda} b)[a_\lambda c]: \\ &\quad + (-1)^{p(a)p(b)} \int_0^\lambda d\mu [b_\mu [a_{\lambda-\mu} c]], \end{aligned} \quad (1.11)$$

which can be derived from (1.4) and (1.8).

1.2. Non-linear Lie conformal algebras

The relationship between Lie conformal algebras and vertex algebras is somewhat similar to the one between Lie algebras and associative algebras. In particular, to any Lie conformal algebra R one canonically associates a vertex algebra $V(R)$ known as the *universal enveloping* vertex algebra of R (see [K, BK, GMS]). In this way one can obtain, for instance, the vertex algebras associated to representations of the *Virasoro* algebra or *affine Kac-Moody* algebras [FZ, K, FB, LL]. These vertex algebras have the property that they are generated by a finite collection of fields whose λ -brackets are *linear* combinations of the same fields and their derivatives. However, there are many examples in which one has a *non-linear* relationship, i.e., the j -th products $a_{(j)}b$ ($j \in \mathbb{Z}_+$) of elements $a, b \in R$ do not necessarily belong to R but are obtained by taking normally ordered products of elements from R . An important class of such examples is provided by *W-algebras*; see [Za, FF, FKW, BS, FB, DSK2] and the references therein.

This has motivated the notion of a *non-linear Lie conformal algebra* [DSK1] as a $\mathbb{C}[T]$ -module R with a λ -bracket $R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes \mathcal{T}(R)$, where $\mathcal{T}(R)$ is the tensor algebra over R , all tensor products being over \mathbb{C} . In order to be able to use induction arguments, one assumes that $R = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_+} R[\Delta]$ is graded so that

$$\Delta(Ta) = \Delta(a) + 1, \quad \Delta(a_{(j)}b) = \Delta(a) + \Delta(b) - j - 1 \quad (1.12)$$

for all $j \in \mathbb{Z}_+$. When $a \in R[\Delta]$ we say that a has *conformal weight* Δ and we use the notation $\Delta(a) = \Delta$. The λ -bracket $[a_\lambda b]$ should then satisfy axioms (1.3) and (1.4). Moreover, in order to impose the Jacobi identity (1.5) we need to extend the λ -bracket to the whole tensor algebra $\mathcal{T}(R)$ (see [DSK1]). We first define the Wick product of $a \in R$ and $B \in \mathcal{T}(R)$ as $:aB: = a \otimes B$, and we then extend the Wick product and λ -bracket to the whole $\mathcal{T}(R)$ using quasi-associativity (1.7) and the Wick formulas (1.8) and (1.11). Then the Jacobi identity (1.5) is imposed modulo the quasi-commutativity relation (1.6). More precisely, following [DSK1]

we introduce the subspace $\mathcal{M}_\Delta(R) \subset \mathcal{T}(R)[\leq \Delta]$ spanned over \mathbb{C} by all elements of the form

$$A \otimes \left((b \otimes c - (-1)^{p(b)p(c)} c \otimes b) \otimes D - \left(\int_{-T}^0 d\lambda [b_\lambda c] \right) D \right),$$

where $b, c \in R$, $A, D \in \mathcal{T}(R)$ and $\Delta(A \otimes b \otimes c \otimes D) \leq \Delta$. Then the Jacobi identity (1.5) must hold modulo $\mathcal{M}_\Delta(R)$ for some $\Delta < \Delta(a) + \Delta(b) + \Delta(c)$.

Recall that a vertex algebra V is *strongly generated* by a subset $\{a^\alpha\} \subset V$ if all monomials $:a^{\alpha_1} \cdots a^{\alpha_s}:$ and $|0\rangle$ linearly span V . Here, as usual, a normally ordered product is taken from right to left, and an empty product is set equal to $|0\rangle$. A vertex algebra V is *freely generated* by an ordered set $\{a^\alpha\} \subset V$ if the monomials

$$:a^{\alpha_1} \cdots a^{\alpha_s}: \quad \text{with } \alpha_i \leq \alpha_{i+1} \text{ and } \alpha_i < \alpha_{i+1} \text{ when } p(a^{\alpha_i}) = \bar{1} \quad (1 \leq i < s),$$

together with $|0\rangle$, form a \mathbb{C} -basis of V .

Consider the subspace $\mathcal{M}(R) = \sum_{\Delta \in \frac{1}{2}\mathbb{Z}_+} \mathcal{M}_\Delta(R)$ of the tensor algebra $\mathcal{T}(R)$. One of the main results of [DSK1] is that the λ -bracket and Wick product are well defined on the quotient $V(R) = \mathcal{T}(R)/\mathcal{M}(R)$ and provide it with the structure of a vertex algebra. Moreover, $V(R)$ is *freely generated* by $R \subset V(R)$, i.e., every ordered \mathbb{C} -basis $\{a^\alpha\}$ of R , compatible with parity and conformal weight, freely generates $V(R)$. Conversely, if a vertex algebra V is freely generated by a free $\mathbb{C}[T]$ -submodule $R \subset V$ graded by a conformal weight, then one can endow R with the structure of a non-linear Lie conformal algebra so that $V \simeq V(R)$ (see [DSK1]).

1.3. Poisson vertex algebras and non-linear Poisson conformal algebras

If we remove the integrals (“quantum corrections”) in the axioms (1.6), (1.7) and (1.8) of a vertex algebra, we arrive at the definition of a *Poisson vertex algebra* (cf. [FB, DLM2]). More precisely, a Poisson vertex algebra is a quintuple $(\mathcal{V}, |0\rangle, T, [\cdot \lambda \cdot], \cdot)$, where $(\mathcal{V}, T, [\cdot \lambda \cdot])$ is a Lie conformal superalgebra, $(\mathcal{V}, |0\rangle, T, \cdot)$ is a unital commutative associative differential superalgebra, and the operations $[\cdot \lambda \cdot]$ and \cdot are related by the *Leibniz rule* (= commutative Wick formula):

$$[a_\lambda(bc)] = [a_\lambda b]c + (-1)^{p(a)p(b)} b[a_\lambda c]. \quad (1.13)$$

Here and below, we write the product $a \cdot b$ as simply ab . As for vertex algebras, in a Poisson vertex algebra \mathcal{V} we can define *n-th products* for every $n \in \mathbb{Z}$ as follows:

$$a_{(-n-1)}b = \frac{1}{n!}(T^n a)b, \quad a_{(n)}b = \partial_\lambda^n [a_\lambda b]|_{\lambda=0}, \quad n \in \mathbb{Z}_+.$$

For example, let $\mathcal{V} = \mathbb{C}[u_0, u_1, \dots]$ be the algebra of polynomials in even indeterminates u_n , let $|0\rangle = 1$ and T be the derivation of \mathcal{V} defined by $Tu_n = u_{n+1}$. Then if we define the λ -bracket on \mathcal{V} by

$$[P_\lambda Q] = \sum_{p, q \in \mathbb{Z}_+} (-1)^p \frac{\partial Q}{\partial u_q} (\lambda + T)^{p+q+1} \frac{\partial P}{\partial u_p},$$

we obtain the so-called *Gardner–Faddeev–Zakharov* Poisson vertex algebra.

Given a Lie conformal algebra R , one canonically associates to it the *universal enveloping* Poisson vertex algebra $\mathcal{V} = \mathcal{S}(R)$, the symmetric algebra over R as a unital commutative associative differential algebra, by extending the λ -bracket of R to $\mathcal{S}(R)$ using the Leibniz rule (1.13) and skew-symmetry (1.4).

However, as for vertex algebras, not all Poisson vertex algebras \mathcal{V} are obtained as enveloping algebras of Lie conformal algebras and, in general, the λ -brackets among generators contain non-linearities. In order to take into account such non-linearities, the notion of a *non-linear Poisson conformal algebra* is then introduced [DSK2]. This is a $\mathbb{C}[T]$ -module R together with a λ -bracket $R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes \mathcal{S}(R)$, extended to $\mathcal{S}(R)$ by the Leibniz rule (1.13), which satisfies sesqui-linearity (1.3), skew-symmetry (1.4) and Jacobi identity (1.5). It is not hard to check, in a way similar (but much easier) to the “quantum” case, that if R is a non-linear Poisson conformal algebra, then $\mathcal{S}(R)$ has an induced Poisson vertex algebra structure. Conversely, if $\mathcal{V} = \mathcal{S}(R)$ is a Poisson vertex algebra freely generated by R , then R has a structure of non-linear Poisson conformal algebra.

1.4. Non-linear deformations of the affine Lie algebra \widehat{sl}_2

Recall from [K] that the affine Lie algebra \widehat{sl}_2 at level $k \in \mathbb{C}$ corresponds to the *current Lie conformal algebra* $R = \mathbb{C}[T]sl_2$, in which the λ -brackets for the standard generators \bar{h}, e, f of sl_2 are:

$$\begin{aligned} [\bar{h}_\lambda \bar{h}] &= 2k\lambda, & [\bar{h}_\lambda e] &= 2e, & [\bar{h}_\lambda f] &= -2f, \\ [e_\lambda f] &= \bar{h} + k\lambda, & [e_\lambda e] &= [f_\lambda f] = 0. \end{aligned} \quad (1.14)$$

We will study “non-linear” vertex algebras and Poisson vertex algebras, which generalize the above “linear” one and which include it as a special case. More precisely, we will keep the above λ -brackets the same, except that we will allow $[e_\lambda f]$ to be an arbitrary polynomial in λ, \bar{h} and its derivatives. It will be convenient to set $\bar{h} = 2h, k = 2\alpha$. We thus restate the problem as follows.

Problem 1.1. Classify all vertex algebras and Poisson vertex algebras strongly generated by three elements h, e, f with λ -brackets

$$\begin{aligned} [h_\lambda h] &= \alpha\lambda|0\rangle, & [h_\lambda e] &= e, & [h_\lambda f] &= -f, & [e_\lambda e] &= [f_\lambda f] = 0, \\ [e_\lambda f] &= :P(\lambda; h, Th, T^2h, \dots):, \end{aligned} \quad (1.15)$$

where $\alpha \in \mathbb{C}$ is an unknown constant and $P \in \mathbb{C}[\lambda, h, Th, T^2h, \dots]$ is an unknown polynomial. We also assume that h is even while the parities of e and f are not fixed beforehand. (In the Poisson case, the normally ordered product is replaced by the commutative associative one.)

In the vertex algebra case, it follows from the commutator formula (1.2) that the field $Y(h, z)$ is a *free boson*:

$$[Y(h, z), Y(h, w)] = \alpha \partial_w \delta(z - w). \quad (1.16)$$

Equivalently, the operators $h_{(n)} \in \text{End } V$ satisfy the commutation relations of the *Heisenberg Lie algebra* with central charge α :

$$[h_{(m)}, h_{(n)}] = \alpha m \delta_{m, -n}, \quad m, n \in \mathbb{Z}.$$

The latter also holds in the Poisson case, due to the Jacobi identity (1.5) and the Leibniz rule (1.13). Moreover, since $h_{(n)}|0\rangle = 0$ for $n \geq 0$, the vacuum vector $|0\rangle$ generates a highest weight module $\langle h \rangle$ over the Heisenberg algebra. For $\alpha \neq 0$, this module is irreducible and is known as the *Fock space*; it is unique up to isomorphism (see e.g. [K, Example 3.5]).

Note that $\langle h \rangle$ can be described as the subalgebra of our (Poisson) vertex algebra generated by h . For $\alpha \neq 0$, the linear map $\mathbb{C}[h, Th, T^2h, \dots] \rightarrow \langle h \rangle$ given by

$$P(h, Th, T^2h, \dots) \mapsto :P(h, Th, T^2h, \dots): \quad (1.17)$$

is a vector space isomorphism, which means that $\langle h \rangle$ is freely generated. The map (1.17) is well defined, because by (1.10) the elements $T^n h$ commute under the normally ordered product. One should note, however, that by (1.7) the normally ordered product in $\langle h \rangle$ is not associative for $\alpha \neq 0$; hence, (1.17) is not an associative algebra isomorphism unless $\alpha = 0$.

If the whole (Poisson) vertex algebra is freely generated by h, e, f , then it is isomorphic to the universal enveloping (Poisson) vertex algebra of a non-linear Lie (respectively, Poisson) conformal algebra $R = \mathbb{C}[T]\langle h, e, f \rangle$ (see [DSK1]). Hence, in the freely generated case Problem 1.1 reduces to the following easier problem.

Problem 1.2. Classify all non-linear Lie conformal algebras and non-linear Poisson conformal algebras with three generators h, e, f with h even and λ -brackets as in (1.15).

Let us explain the difference between Problems 1.1 and 1.2. First, if R is a non-linear Lie conformal algebra solving Problem 1.2, then its universal enveloping vertex algebra $V(R)$ is freely generated and solves Problem 1.1 with the same α and P . The Jacobi identities (1.5) for $V(R)$ can be deduced from the λ -brackets (1.15) and the other axioms of a vertex algebra. Moreover, by [DSK1], any other vertex algebra V solving Problem 1.1 for the same α and P is a quotient of $V(R)$. On the other hand, for certain α and P there exist vertex algebras solving Problem 1.1, which are not quotients of freely generated ones (see Section 2 below). These algebras satisfy the Jacobi identities (1.5) only because of some additional relations among the generators h, e, f and their derivatives. In this case, there is no corresponding non-linear Lie conformal algebra solving Problem 1.2. Similar remarks apply in the Poisson case.

1.5. The vertex algebras $V_{\mathbb{Z}\sqrt{\beta}}$, V_{-1}^d and $\mathcal{W}_n^{(2)}$

In the present paper we solve Problem 1.1 for $\alpha \neq 0$ and Problem 1.2 for arbitrary α . Our main classification results are stated in Section 2 below; here we just mention a few examples.

The first example is provided by the *lattice vertex algebras* $V_{\mathbb{Z}\sqrt{\beta}}$, where $\beta \in \mathbb{N}$ (see [B1, FLM, K, FB, LL]). These algebras solve Problem 1.1 with $\alpha = 1/\beta$ and a homogeneous polynomial P of degree $d = \beta - 1$ for the grading given by $\deg \lambda = \deg T = \deg h = 1$. They are discussed in detail in Section 2.1 below.

Another series of vertex algebras solving Problem 1.1 consists of the algebras V_{-1}^d constructed in the present paper. They correspond to $\alpha = -1$ and an explicit polynomial P of degree d (see Sections 2.2, 2.3 and 2.4 below). Since V_{-1}^d is freely generated by h, e, f , it is the universal enveloping vertex algebra of a non-linear Lie conformal algebra R_{-1}^d solving Problem 1.2.

In [FS], Feigin and Semikhatov introduced a sequence of vertex algebras $\mathcal{W}_n^{(2)}$ ($n \in \mathbb{N}$). These algebras involve a certain parameter $k \in \mathbb{C}$ (similar to the level of the affine algebra) and are only defined for $k \neq -n$, because the Virasoro central charge has a pole at $k = -n$ (see [FS, Eq. (1.1)]). For $n = 2$, one obtains the *affine vertex algebra* of sl_2 at level k , while $\mathcal{W}_3^{(2)}$ coincides with the *Bershadsky–Polyakov $W_3^{(2)}$ algebra*. The algebras $\mathcal{W}_n^{(2)}$ do not solve Problem 1.1, because in them the λ -bracket $[e_\lambda f]$ is a polynomial not only of h and its derivatives but also of other elements, which in particular include the Virasoro one. After comparing our formulas with those in [FS, Appendix A], one sees that for $n \leq 4$ our vertex algebra V_{-1}^{n-1} can be obtained as a certain subquotient of $\mathcal{W}_n^{(2)}$ at the *critical level* $k = -n$. We believe this is true for all n ; however, in general it cannot be done explicitly because the methods of [FS] are very different from ours.

1.6. Plan of the paper

In Section 2, we state the classification of all vertex algebras solving Problem 1.1 for $\alpha \neq 0$ and all non-linear Lie conformal algebras solving Problem 1.2 for arbitrary α . The Poisson case is discussed as well.

In Section 3, we make some preliminary observations based on grading and change of basis, which reduce Problems 1.1 and 1.2 to two equations: the Jacobi identity for elements h, e, f and the Jacobi identity for e, e, f . The polynomial $P(\lambda; x_1, x_2, \dots)$ can be assumed homogeneous of degree d , where $\deg \lambda = 1$, $\deg x_k = k$.

We then impose, in Section 4, the Jacobi identity for h, e, f . This determines for every degree d the polynomial P explicitly, provided that $\alpha \neq 0$. For $\alpha = 0$, the condition is instead that the polynomial P is independent of λ . The treatment is the same, in that section, both for vertex algebras and vertex Poisson algebras.

In Section 5, we impose the Jacobi identity for elements e, e, f in a vertex algebra solving Problem 1.1 with $\alpha \neq 0$. This determines α . We show that there are two solutions for every degree d , corresponding to $\alpha = -1$ (giving the freely generated V_{-1}^d) and $\alpha = 1/(d+1)$ (giving the lattice vertex algebra $V_{\mathbb{Z}\sqrt{d+1}}$).

Next, in Section 6, we consider the case of a non-linear Lie conformal algebra with $\alpha = 0$. We prove that the only possibility is the current Lie conformal algebra $R = \mathbb{C}[T]sl_2$, thus completing the solution of Problem 1.1 (for $\alpha \neq 0$) and Problem 1.2 (for any α) in the “quantum” case.

In Section 7, we consider the Poisson, or “classical,” case. The treatment is similar to that of the previous two sections, but much simpler. Besides the current Lie conformal algebra $R = \mathbb{C}[T]sl_2$ and its universal enveloping Poisson vertex algebra, the only solutions we obtain are with $\alpha = 0$ and $P = h^d$. This provides examples of freely generated Poisson vertex algebras, which are not “semiclassical” limits of any vertex algebra.

In Section 8, we construct free-field realizations of (certain quotients of) the vertex algebras V_{-1}^d solving Problem 1.1 with $\alpha = -1$, which generalize the Wakimoto realization of the affine vertex algebra of sl_2 at the critical level -2 .

In Section 9, we determine the Zhu algebra of V_{-1}^d , which turns out to be one of the associative algebras introduced by Smith in [S]. The same method is also used to find the Zhu algebra of the lattice vertex algebra $V_{\mathbb{Z}\sqrt{d+1}}$, thus reproducing a result of [DLM1].

2. Classification results

In this section, we state the classification of all algebras solving Problem 1.1 for $\alpha \neq 0$ and Problem 1.2 for arbitrary α . First, let us recall a well-known important example, which provided an early indication that the problems are interesting.

2.1. Lattice vertex algebras of rank one

Fix a positive integer β , and consider the rank one lattice $\mathbb{Z}\sqrt{\beta} \subset \mathbb{R}$ and the corresponding *lattice vertex algebra* $V_{\mathbb{Z}\sqrt{\beta}}$ (see [B1, FLM, K, FB, LL]). Let us recall the definition and properties of $V_{\mathbb{Z}\sqrt{\beta}}$, following Sections 5.4 and 5.5 in [K]. As a vector space,

$$V_{\mathbb{Z}\sqrt{\beta}} = \mathbb{C}[q, q^{-1}; h_{(-1)}, h_{(-2)}, h_{(-3)}, \dots] \simeq \langle h \rangle \otimes \mathbb{C}[q, q^{-1}],$$

where $\langle h \rangle = \mathbb{C}[h_{(-1)}, h_{(-2)}, \dots]$ is the Fock space for the free boson $Y(h, z)$ with $\alpha = 1/\beta$ (see (1.16)). We let $h_{(n)}$ act trivially on q^k for $n > 0$, while $h_{(0)}$ acts on $V_{\mathbb{Z}\sqrt{\beta}}$ as $q\partial_q$. This means that each q^k ($k \in \mathbb{Z}$) is a highest weight vector for the Heisenberg algebra; in particular, the vacuum vector is $|0\rangle = 1$. The parity of q^k is $k\beta \bmod 2\mathbb{Z}$.

The vertex algebra $V_{\mathbb{Z}\sqrt{\beta}}$ is generated by the free boson $Y(h, z)$ and by the following quantum fields known as vertex operators ($k \in \mathbb{Z}$):

$$Y(q^k, z) = q^k z^{k\beta q\partial_q} \exp\left(-\sum_{n<0} \frac{z^{-n}}{n} k\beta h_{(n)}\right) \exp\left(-\sum_{n>0} \frac{z^{-n}}{n} k\beta h_{(n)}\right). \quad (2.1)$$

In fact, $V_{\mathbb{Z}\sqrt{\beta}}$ is strongly generated by the elements h , $e = q$, and $f = q^{-1}$. The element h is even, while the parities of e and f are both equal to $\beta \bmod 2\mathbb{Z}$. The λ -brackets among the generators h, e, f are given by (1.15), where $\alpha = 1/\beta$ and (cf. [K, Eq. (5.5.18)])

$$P(\lambda; h, Th, T^2h, \dots) = \sum_{n=0}^{\beta-1} \frac{\lambda^n}{n!} S_{\beta-1-n} \left(\beta \frac{h}{1!}, \beta \frac{Th}{2!}, \beta \frac{T^2h}{3!}, \dots \right). \quad (2.2)$$

Here and further,

$$S_n(x_1, x_2, x_3, \dots) = \sum_{\substack{i_1+2i_2+3i_3+\dots=n \\ i_s \in \mathbb{Z}_+}} \frac{x_1^{i_1}}{i_1!} \frac{x_2^{i_2}}{i_2!} \frac{x_3^{i_3}}{i_3!} \dots \quad (2.3)$$

denotes the *elementary Schur polynomial* of degree n (which is homogeneous when we set $\deg x_k = k$).

Therefore, the vertex algebra $V_{\mathbb{Z}\sqrt{\beta}}$ provides a solution of Problem 1.1 with $\alpha = 1/\beta$, $\beta \in \mathbb{N}$. However, this vertex algebra is *not* freely generated by h, e, f because of the additional relations

$$Te = \beta :he:, \quad Tf = -\beta :hf:, \quad (2.4)$$

which follow from (2.1) and the translation covariance axiom. Finally, note that $V_{\mathbb{Z}\sqrt{\beta}}$ is a *simple* vertex algebra, i.e., it does not have nontrivial proper ideals.

2.2. Non-linear Lie conformal algebras R_{-1}^d

Consider the ring $\mathbb{C}[\lambda, h, Th, T^2h, \dots]$, equipped with the derivation T such that $T\lambda = 0$. For an arbitrary polynomial $p(\lambda) \in \mathbb{C}[\lambda]$, we let

$$P(\lambda; h, Th, \dots) = p(\lambda + T - h) 1 \in \mathbb{C}[\lambda, h, Th, \dots], \quad (2.5)$$

where $p(\lambda + T - h)$ is expanded by letting T act on its right and by letting $T1 = 0$. So for example,

$$(\lambda + T - h)^2 1 = (\lambda + T - h)(\lambda - h) = \lambda^2 - 2\lambda h - Th + h^2.$$

When we consider P under the normally ordered product, it becomes a polynomial in λ with coefficients belonging to a non-linear Lie conformal algebra or a vertex algebra. In particular, we let $:1: = |0\rangle$ be the vacuum vector.

Proposition 2.1. *For every polynomial $p \in \mathbb{C}[\lambda]$, there exists a non-linear Lie conformal algebra $R = \mathbb{C}[T]\langle h, e, f \rangle$ with the λ -brackets (1.15), where h, e and f are even, $\alpha = -1$ and $P(\lambda; h, Th, \dots) = p(\lambda + T - h)1$.*

The proof will be given in Section 5.1 below.

We will show in Section 3.1 that, without loss of generality, one can assume that the polynomial P in Problem 1.2 is homogeneous for the grading given by $\deg \lambda = \deg T = \deg h = 1$. If P is homogeneous of degree d and is given by (2.5), then up to rescaling $p(\lambda) = \lambda^d$. In this case, we will denote the non-linear Lie conformal algebra R from the proposition as R_{-1}^d . Note that $d = 1$ corresponds to $[e_\lambda f] = \lambda - h$, which after rescaling gives the current Lie conformal algebra $\mathbb{C}[T]sl_2$ at the critical level $k = -2$ (see (1.14)).

2.3. Solution of Problem 1.2 in the “quantum” case.

The following theorem, which will be proved in Sections 5.3 and 6.2 below, is one of the main results of the paper.

Theorem 2.2. *A complete classification of non-linear Lie conformal algebras $R = \mathbb{C}[T]\langle h, e, f \rangle$ with h even and with λ -bracket as in (1.15) is the following. Assume, without loss of generality, that the polynomial P in (1.15) is homogeneous of degree $d \geq 1$ (with respect to the grading given by $\deg \lambda = \deg T = \deg h = 1$). Then:*

- (a) *When $d = 1$, α is arbitrary, and R is isomorphic to the current Lie conformal algebra $\mathbb{C}[T]sl_2$ at level $k = 2\alpha$, i.e., after rescaling*

$$[e_\lambda f] = h + \alpha \lambda.$$

- (b) *When $d \geq 2$, $\alpha = -1$, and R is isomorphic to the non-linear Lie conformal algebra R_{-1}^d , i.e., e, f are even and*

$$[e_\lambda f] = :(\lambda + T - h)^d 1: . \quad (2.6)$$

In Section 4, we will provide another formula for the λ -bracket in R_{-1}^d , which involves the elementary Schur polynomials (2.3) and is similar to formula (2.2) above.

2.4. Solution of Problem 1.1 in the “quantum” case for $\alpha \neq 0$.

The following result describes all vertex algebras solving Problem 1.1 with $\alpha \neq 0$ and a homogeneous polynomial P .

Theorem 2.3. *Consider vertex algebras strongly generated by elements h, e, f with λ -bracket relations as in (1.15), where $\alpha \neq 0$ and the polynomial P is homogeneous of degree $d \geq 1$ (when $\deg \lambda = \deg T = \deg h = 1$). Then, up to isomorphism, a complete list of such vertex algebras V is:*

- (a) *When $d = 1$, α is arbitrary, and V is a quotient of the universal enveloping vertex algebra of the current Lie conformal algebra $\mathbb{C}[T]sl_2$ at level $k = 2\alpha$.*
 (b) *When $d \geq 2$, $\alpha = -1$, and V is a quotient of the universal enveloping vertex algebra $V_{-1}^d := V(R_{-1}^d)$ of the non-linear Lie conformal algebra R_{-1}^d .*
 (c) *When $d \geq 2$, $\alpha = 1/(d+1)$, and V is isomorphic to the lattice vertex algebra $V_{\mathbb{Z}\sqrt{d+1}}$. In this case, e and f must have the same parity as $d+1$.*

We will prove this theorem in Section 5.3 below. The arguments in the proof do not work for $\alpha = 0$, and in fact they may be used to construct counterexamples.

Notice that, by the Frenkel–Kac construction [FrK, K], the lattice vertex algebra $V_{\mathbb{Z}\sqrt{2}}$ is a quotient of the universal enveloping vertex algebra of the current Lie conformal algebra $\mathbb{C}[T]sl_2$ at level $k = 1$, which corresponds to $\alpha = 1/2$. Thus, for $d = 1$, part (c) of Theorem 2.3 is included in part (a).

2.5. Solution of Problems 1.1 and 1.2 in the Poisson case

It is easy to check that, for any polynomial $p(\lambda) \in \mathbb{C}[\lambda]$, the λ -bracket

$$[e_\lambda f] = p(h),$$

together with the other formulas in (1.15) with $\alpha = 0$, provides an example of a non-linear Poisson conformal algebra solving Problem 1.2. The following two theorems will be proved in Section 7.

Theorem 2.4. *A complete classification of non-linear Poisson conformal algebras $R = \mathbb{C}[T]\langle e, f, h \rangle$ with h even and λ -bracket as in (1.15) is the following:*

- (a) *The current Lie conformal algebra $R = \mathbb{C}[T]sl_2$ at level $k = 2\alpha$. In this case, e and f are even and the polynomial P is homogeneous of degree 1.*
- (b) *For $\alpha = 0$, both e and f are even, and $P = p(h)$ is an arbitrary polynomial of h .*

Theorem 2.5. *Let \mathcal{V} be a Poisson vertex algebra strongly generated by elements $h, e, f \in \mathcal{V}$ with λ -bracket relations as in (1.15) with $\alpha \neq 0$. Then \mathcal{V} is a quotient of the universal enveloping Poisson vertex algebra $\mathcal{S}(R)$ of the current Lie conformal algebra $R = \mathbb{C}[T]sl_2$ at level $k = 2\alpha$.*

3. Preliminary observations

First of all, we note that (1.15) does not provide explicitly all possible λ -brackets among the generators h, e, f , but the remaining ones are determined by the skew-symmetry axiom (1.4):

$$[e_\lambda h] = -e, \quad [f_\lambda h] = f, \quad [f_\lambda e] = -(-1)^{p(e)p(f)} :P(-\lambda - T; h, Th, T^2h, \dots):.$$

Then the sesqui-linearity (1.3) is used to extend the λ -bracket to $\mathbb{C}[T]\langle h, e, f \rangle$. Hence, the sesqui-linearity and skew-symmetry are automatically satisfied, and we only need to impose the Jacobi identities (1.5) involving the generators h, e and f . Furthermore, the only Jacobi identities that are not trivially satisfied are the ones for the triples (h, e, f) , (e, e, f) , and (f, f, e) .

3.1. Grading and symmetry conditions

In order to solve Problem 1.2, we need to find all α and P such that the λ -bracket given by (1.15) satisfy the Jacobi identities for the triples (h, e, f) , (e, e, f) , and (f, f, e) . These identities lead to certain equations, which are linear in P . Therefore, for fixed $\alpha \in \mathbb{C}$, the set of all polynomials P solving Problem 1.2 forms a complex vector space. (This conclusion does *not* hold for the set of polynomials P solving Problem 1.1, because each particular solution may involve its own set of additional relations among the generators; see the discussion at the end of Section 1.4.)

Hence, for Problem 1.2, we can assume that P is a *homogeneous* polynomial of degree d for the grading given by $\deg \lambda = \deg T = \deg h = 1$. Then the λ -bracket relations (1.15) are homogeneous as well. This grading is compatible with a grading by *conformal weight* in such a way that (cf. (1.12)):

$$\Delta(h) = 1, \quad \Delta(e) + \Delta(f) = d + 1. \quad (3.1)$$

We are going to make this assumption for Problem 1.1 as well. Notice also that, by replacing the generator e by γe ($\gamma \in \mathbb{C}$), one obtains another solution with the polynomial P replaced by γP .

Next, it follows from the skew-symmetry (1.4) that relations (1.15) are invariant under the change of generators

$$\tilde{h} = -h, \quad \tilde{e} = f, \quad \tilde{f} = e, \quad (3.2)$$

if we replace the polynomial P by

$$\tilde{P}(\lambda; h, Th, T^2h, \dots) = -(-1)^{p(e)p(f)} P(-\lambda - T; -h, -Th, \dots). \quad (3.3)$$

Hence, if $R = \mathbb{C}[T]\langle h, e, f \rangle$ is a non-linear Lie conformal algebra satisfying the assumptions of Problem 1.2 for a given choice of α and P , then $\tilde{R} = \mathbb{C}[T]\langle \tilde{f}, \tilde{h}, \tilde{e} \rangle$ is also a non-linear Lie conformal algebra of the same type, with $\tilde{\alpha} = \alpha$ and $\tilde{P}(\lambda; h, Th, \dots)$ as in (3.3). Conversely, if the Jacobi identities for the triples (h, e, f) and (e, e, f) hold with both polynomials P and \tilde{P} , then the Jacobi identity for the triple (f, f, e) follows automatically. In particular, if P satisfies the *symmetry condition* $\tilde{P} = \pm P$, then the Jacobi identity for the triple (f, f, e) follows from the one for (e, e, f) . In fact, we will show in Section 4 below that for $\alpha \neq 0$ this symmetry condition follows from the Jacobi identity for (h, e, f) .

In conclusion, for both Problem 1.1 and 1.2, we will assume that P is homogeneous of degree d , and if $\tilde{P} = \pm P$, we only need to impose the Jacobi identities for the triples (h, e, f) and (e, e, f) . This will be done separately in the following sections.

3.2. Technical results

Here, we collect several computational results which will be useful in the sequel. Throughout this subsection β will be a fixed complex number. For a formal series $\varphi(x) \in \mathbb{C}[[x, x^{-1}]]$, we will denote by $\text{Reg}_x \varphi(x)$ its *regular part*, i.e.,

$$\text{Reg}_x x^n = \begin{cases} x^n, & \text{if } n \geq 0, \\ 0, & \text{if } n < 0. \end{cases}$$

Let Δ_x be the *difference operator* acting on regular series by

$$(\Delta_x \varphi)(x) = \frac{\varphi(x) - \varphi(0)}{x}, \quad \varphi(x) \in \mathbb{C}[[x]].$$

Lemma 3.1. *We have:*

$$\Delta_x \text{Reg}_x \varphi(x) = \text{Reg}_x x^{-1} \varphi(x), \quad \varphi(x) \in \mathbb{C}[[x, x^{-1}]].$$

Proof. It suffices to check this equation for $\varphi(x) = x^n$ ($n \in \mathbb{Z}$), in which case it is straightforward. \square

We introduce the hypergeometric function

$$\Phi_\beta(x) = \sum_{n=0}^{\infty} \binom{\beta}{n} \frac{x^n}{n!},$$

where, as usual, the binomial coefficient is given by $\binom{\beta}{n} = \frac{\beta(\beta-1)\cdots(\beta-n+1)}{n!}$. Using that $(n+1)\binom{\beta}{n+1} = (\beta-n)\binom{\beta}{n}$, it is easy to check that $\Phi_\beta(x)$ satisfies the differential equation

$$(x\partial_x^2 + (x+1)\partial_x - \beta)\Phi_\beta(x) = 0. \quad (3.4)$$

Let us also introduce the formal series

$$\Psi_\beta(x, y) = \partial_y \Phi_\beta(-xy) = \sum_{n=1}^{\infty} \binom{\beta}{n} \frac{(-x)^n y^{n-1}}{(n-1)!}. \quad (3.5)$$

Proposition 3.2. *Let a and b be two elements in a vertex algebra satisfying $[b_\lambda b] \in \mathbb{C}[\lambda]|0\rangle$ and $[b_\lambda a] = \beta a$, where b is even. Then the following identities hold:*

$$:(e^{yT}a)(e^{x(T+b)}1): = \text{Reg}_y(1 - xy^{-1})^\beta : (e^{x(T+b)}1)(e^{yT}a):, \quad (3.6)$$

$$:a(e^{x(T+b)}1): = : (e^{x(T+b)}1)(\Phi_\beta(-xT)a):, \quad (3.7)$$

$$[a_\lambda : (e^{x(T+b)}1):] = : (e^{x(T+b)}1)(\Psi_\beta(x, \lambda + T)a):, \quad (3.8)$$

where x and y are formal variables and $\beta \in \mathbb{C}$. In the right-hand side of (3.6), the function $(1 - xy^{-1})^\beta$ is expanded as a power series of x, y in the domain $|x| < |y|$.

Recall that a normally ordered product of more than two factors is taken from right to left (cf. (1.9)), and that $T1 = 0$. In the right-hand sides of the above equations, $e^{x(T+b)}1$ is considered as a formal power series in x whose coefficients are polynomials in b, Tb, T^2b , etc. The latter are multiplied by elements of the form $T^k a$, and give elements of the vertex algebra only after taking the normally ordered product. This is well defined because, by (1.10), the condition $[b_\lambda b] \in \mathbb{C}[\lambda]|0\rangle$ implies that all elements $T^k b$ ($k \in \mathbb{Z}_+$) commute under the normally ordered product.

So, for example, we have

$$:((T+b)^2 1)a: = :(Tb)a: + :b(:ba):,$$

which differs from

$$:(: (T+b)^2 1:)a: = :(Tb)a: + :(:bb:)a:,$$

or from

$$:(T+b)^2 a: = :(Tb)a: + :b(:ba): + 2:b(Ta): + T^2 a.$$

Proof of Proposition 3.2. To prove (3.6), we notice that both sides are equal to $e^{yT}a$ for $x = 0$, and we are going to show they both satisfy the same first-order differential equation in x . Denote the left-hand side by $A(x, y)$. Using (1.10) and the fact that T is a derivation of the normally ordered product, we find:

$$\begin{aligned} (\partial_x + \partial_y - T)A(x, y) &= :(e^{yT}a)b(e^{x(T+b)}1): \\ &= :b(e^{yT}a)(e^{x(T+b)}1): + : \left(\int_{-T}^0 d\lambda [(e^{yT}a)_\lambda b] \right) (e^{x(T+b)}1):. \end{aligned}$$

Then, by the sesqui-linearity (1.3) and skew-symmetry (1.4),

$$\int_{-T}^0 d\lambda [(e^{yT}a)_\lambda b] = \int_{-T}^0 d\lambda e^{-y\lambda} [a_\lambda b] = - \int_{-T}^0 d\lambda e^{-y\lambda} \beta a = -\beta \frac{e^{yT} - 1}{y} a.$$

Putting these together, we obtain:

$$(\partial_x + \partial_y - T)A(x, y) = :bA(x, y): - \beta \Delta_y A(x, y).$$

The right-hand side of (3.6) satisfies the same differential equation, due to Lemma 3.1 and the fact that

$$(\partial_x + \partial_y)(1 - xy^{-1})^\beta = -\beta y^{-1}(1 - xy^{-1})^\beta.$$

This proves (3.6).

To derive (3.7), we set $y = 0$ in both sides of formula (3.6). Using the binomial expansion in the domain $|x| < |y|$, it is straightforward to compute

$$\left(\text{Reg}_y (1 - xy^{-1})^\beta e^{yT} \right) \Big|_{y=0} = \sum_{n=0}^{\infty} \binom{\beta}{n} (-x)^n \frac{T^n}{n!} = \Phi_\beta(-xT).$$

To prove (3.8), we are going to follow the same strategy as above. We first note that both sides vanish for $x = 0$. Denoting the left-hand side by $B(x, \lambda)$, we compute

$$\partial_x B(x, \lambda) = [a_\lambda : (T + b)(e^{x(T+b)} 1) :] = (\lambda + T)B(x, \lambda) + [a_\lambda : b(e^{x(T+b)} 1) :],$$

using the sesqui-linearity (1.3). By the noncommutative Wick formula (1.8) and $[a_\lambda b] = -\beta a$, we have:

$$[a_\lambda : b(e^{x(T+b)} 1) :] = -\beta : a(e^{x(T+b)} 1) : + : b B(x, \lambda) : - \beta \int_0^\lambda d\mu B(x, \mu).$$

Therefore, $B(x, \lambda)$ satisfies the following differential equation in x :

$$\partial_x B(x, \lambda) = :(\lambda + T + b) B(x, \lambda) : - \beta \int_0^\lambda d\mu B(x, \mu) - \beta : (e^{x(T+b)} 1) (\Phi_\beta(-xT) a) :.$$

For the last term we used (3.7). To show that the right-hand side of (3.8) satisfies the same differential equation, it suffices to check that

$$\partial_x \Psi_\beta(x, \lambda + T) = (\lambda + T) \Psi_\beta(x, \lambda + T) - \beta \int_0^\lambda d\mu \Psi_\beta(x, \mu + T) - \beta \Phi_\beta(-xT).$$

Using that $\Psi_\beta(x, \lambda + T) = \partial_\lambda \Phi_\beta(-x(\lambda + T))$, it is easy to reduce this identity to

$$\partial_x \partial_\mu \Phi_\beta(-x\mu) = \mu \partial_\mu \Phi_\beta(-x\mu) - \beta \Phi_\beta(-x\mu),$$

which holds thanks to equation (3.4). This completes the proof. \square

Proposition 3.2 takes a particularly nice form when $\beta = \pm 1$, since

$$\Phi_1(x) = 1 + x, \quad \Psi_1(x, y) = -x, \quad \Phi_{-1}(x) = e^{-x}, \quad \Psi_{-1}(x, y) = xe^{xy}. \quad (3.9)$$

Corollary 3.3. *Under the assumptions of Proposition 3.2, for any polynomial $p(\lambda) \in \mathbb{C}[\lambda]$, we have:*

(a) For $\beta = 1$,

$$\begin{aligned} :a(p(T+b)1): &= : (p(T+b)1)a : - : (p'(T+b)1)(Ta) :, \\ [a_\lambda : (p(T+b)1) :] &= - : (p'(T+b)1)a :, \end{aligned}$$

(b) For $\beta = -1$,

$$\begin{aligned} :a(p(T+b)1): &= :p(T+b)a:, \\ [a_\lambda:(p(T+b)1):] &= :(p'(\lambda+T+b)1)a:, \end{aligned}$$

where p' denotes the derivative of p .

One important difference between the above formulas is that in the case $\beta = 1$ the translation operator T gives 0 when applied to 1 and it does not carry to the element a , while in the case $\beta = -1$ the operator T is applied to a (recall the observations after Proposition 3.2).

Proof of Corollary 3.3. By (3.8) and (3.9), we have in the case $\beta = 1$:

$$\begin{aligned} :a(e^{x(T+b)}1): &= :(e^{x(T+b)}1)(a - xTa):, \\ [a_\lambda:(e^{x(T+b)}1):] &= -x:(e^{x(T+b)}1)a:. \end{aligned}$$

Similarly, in the case $\beta = -1$:

$$\begin{aligned} :a(e^{x(T+b)}1): &= :e^{x(T+b)}a:, \\ [a_\lambda:(e^{x(T+b)}1):] &= x:e^{x(\lambda+T+b)}a:, \end{aligned}$$

using that T is a derivation of the normally ordered product. The corollary now follows by comparing the coefficients of the various powers of x and by linearity. \square

4. Jacobi identity for h, e, f

Throughout this section, we will work in a vertex algebra or a Poisson vertex algebra solving Problem 1.1. We will consider the Jacobi identity for the triple (h, e, f) :

$$[h_\lambda[e_\mu f]] = [[h_\lambda e]_{\lambda+\mu} f] + [e_\mu[h_\lambda f]] \quad (4.1)$$

as an equation for the polynomial P . We will show that for $\alpha \neq 0$ this equation determines P uniquely up to a multiplicative constant for each degree d .

4.1. Equations for the polynomial P

By (1.15), the right-hand side of (4.1) is simply

$$[e_{\lambda+\mu} f] - [e_\mu f] = :P(\lambda + \mu; h, Th, T^2h, \dots): - :P(\mu; h, Th, T^2h, \dots):.$$

To compute the left-hand side of (4.1), we observe that the Wick formula (1.8) when applied to polynomials of $T^k h$ reduces to its *commutative* version (1.13), because h is a free field, i.e., $[h_\lambda h]$ is annihilated by T . We obtain that

$$\begin{aligned} [h_\lambda :P(\mu; h, Th, T^2h, \dots):] &= \sum_{k=1}^{\infty} [h_\lambda(T^{k-1}h)] : \frac{\partial P}{\partial x_k}(\mu; h, Th, T^2h, \dots): \\ &= \sum_{k=1}^{\infty} \alpha \lambda^k : \frac{\partial P}{\partial x_k}(\mu; h, Th, T^2h, \dots):, \end{aligned}$$

where $x_k = T^{k-1}h$ ($k \in \mathbb{N}$). The above formulas remain the same in the Poisson case if we replace the normally ordered product by the commutative associative one.

Since the map (1.17) is injective, we can ignore the normal orderings in the above two equations; then (4.1) becomes equivalent to the following identity in the polynomial ring $\mathbb{C}[\lambda, \mu, x_1, x_2, \dots]$:

$$P(\lambda + \mu; x_1, x_2, \dots) - P(\mu; x_1, x_2, \dots) = \sum_{k=1}^{\infty} \alpha \lambda^k \frac{\partial P}{\partial x_k}(\mu; x_1, x_2, \dots).$$

By Taylor's formula, this identity is equivalent to the system of equations

$$\frac{1}{k!} \frac{\partial^k P}{\partial \lambda^k}(\lambda; x_1, x_2, \dots) = \alpha \frac{\partial P}{\partial x_k}(\lambda; x_1, x_2, \dots), \quad k = 1, 2, \dots \quad (4.2)$$

4.2. Solving the equations

For $\alpha = 0$, equations (4.2) hold if and only if $P(\lambda; x_1, x_2, \dots)$ is independent of λ . Until the end of this section, we will assume that $\alpha \neq 0$ and we set $\beta = 1/\alpha$.

Equations (4.2) are homogeneous if we define a grading by $\deg \lambda = 1$, $\deg x_k = k$. After the substitution $x_k = T^{k-1}h$, this grading agrees with the one defined earlier (in Section 3.1) by $\deg \lambda = \deg T = \deg h = 1$. Henceforth, we will assume that the polynomial P is homogeneous of degree $d \geq 1$.

Lemma 4.1. *For $\alpha \neq 0$, every solution P of equations (4.2), homogeneous of degree d with respect to the grading $\deg \lambda = 1$, $\deg x_k = k$, is of the form*

$$P(\lambda; x_1, x_2, x_3, \dots) = \gamma S_d\left(\lambda + \beta \frac{x_1}{1!}, \beta \frac{x_2}{2!}, \beta \frac{x_3}{3!}, \dots\right), \quad (4.3)$$

where $\beta = 1/\alpha$, $\gamma \in \mathbb{C}$, and S_d is the elementary Schur polynomial defined by (2.3).

Proof. After making the change of variables

$$y_k = \beta \frac{x_k}{k!}, \quad x_k = \alpha k! y_k, \quad k = 1, 2, \dots,$$

equations (4.2) can be rewritten as follows:

$$\frac{\partial^k P}{\partial \lambda^k} = \frac{\partial P}{\partial y_k}, \quad k = 1, 2, \dots$$

For $k = 1$, this implies that

$$P(\lambda; x_1, x_2, \dots) = Q(\lambda + y_1, y_2, y_3, \dots)$$

for some polynomial Q . The rest of the equations are then equivalent to:

$$\frac{\partial Q}{\partial y_k}(y_1, y_2, \dots) = \frac{\partial^k Q}{\partial y_1^k}(y_1, y_2, \dots), \quad k = 1, 2, \dots \quad (4.4)$$

If the coefficient of y_d in $Q(y_1, y_2, \dots)$ is equal to γ , then

$$\frac{\partial^{i_1}}{\partial y_1^{i_1}} \frac{\partial^{i_2}}{\partial y_2^{i_2}} \cdots \frac{\partial^{i_d}}{\partial y_d^{i_d}} Q = \frac{\partial Q}{\partial y_d} = \gamma,$$

for any choice of $i_s \in \mathbb{Z}_+$ such that $1i_1 + 2i_2 + \cdots + di_d = d$. Therefore, the coefficient of $y_1^{i_1} \cdots y_d^{i_d}$ in $Q(y_1, y_2, \dots)$ is equal to $\gamma/i_1! \cdots i_d!$, and hence $Q = \gamma S_d$. \square

Recall that the generating function of the elementary Schur polynomials is

$$\sum_{n=0}^{\infty} z^n S_n(y_1, y_2, \dots) = \exp\left(\sum_{k=1}^{\infty} z^k y_k\right), \quad (4.5)$$

where $S_0 \equiv 1$. Indeed, it is obvious that the function $\exp(\sum z^k y_k)$ satisfies equations (4.4). From (4.3) and (4.5), we obtain

$$\begin{aligned} P(\lambda; h, Th, \dots) &= \gamma \operatorname{Res}_z z^{-d-1} \exp\left(z\lambda + \sum_{k=1}^{\infty} z^k \beta \frac{T^{k-1}h}{k!}\right) \\ &= \gamma \operatorname{Res}_z z^{-d-1} e^{z\lambda} \exp\left(\frac{e^{zT} - 1}{T} \beta h\right). \end{aligned} \quad (4.6)$$

Remark 4.2. Expanding the exponential $e^{z\lambda}$ in (4.6) and taking the residue, we get

$$P(\lambda; h, Th, T^2h, \dots) = \gamma \sum_{n=0}^d \frac{\lambda^n}{n!} S_{d-n}\left(\beta \frac{h}{1!}, \beta \frac{Th}{2!}, \beta \frac{T^2h}{3!}, \dots\right).$$

This is exactly formula (2.2) for $\beta = d + 1$ and $\gamma = 1$. Conversely, (2.2) can be rewritten as (4.3) with $\beta = d + 1$.

4.3. Symmetry condition

Now we will show that the polynomial P in Lemma 4.1 satisfies the symmetry condition of Section 3.1.

Lemma 4.3. *The polynomial P , given by (4.3), satisfies*

$$P(-\lambda - T; h, Th, \dots) = (-1)^d P(\lambda; h, Th, \dots).$$

Equivalently, we have $\tilde{P} = (-1)^{d+1+p(e)p(f)} P$, where \tilde{P} is defined by (3.3).

Proof. Using (4.6) and the fact that T is a derivation, we compute:

$$\begin{aligned} P(-\lambda - T; -h, -Th, \dots) &= \gamma \operatorname{Res}_z z^{-d-1} e^{z(-\lambda-T)} \exp\left(\frac{e^{zT} - 1}{T} \beta(-h)\right) \\ &= \gamma \operatorname{Res}_z z^{-d-1} e^{-z\lambda} \exp\left(\frac{e^{-zT} - 1}{T} \beta h\right) \\ &= (-1)^d P(\lambda; h, Th, \dots), \end{aligned}$$

as claimed. \square

4.4. Another formula for the polynomial P

Equation (4.6) can also be written as

$$P(\lambda; h, Th, \dots) = \gamma \operatorname{Res}_z z^{-d-1} e^{z\lambda} \exp\left(\beta \int_0^z dx e^{xT} h\right).$$

Now we will give a formula equivalent to it.

Lemma 4.4. *The following identity holds in the ring $\mathbb{C}[h, Th, T^2h, \dots][[z]]$:*

$$\exp\left(\beta \int_0^z dx e^{xT} h\right) = e^{z(T+\beta h)} 1,$$

where $\beta \in \mathbb{C}$ and in the right-hand side $T1 = 0$.

Proof. Denote the left-hand side of this equation by $A(z)$. Since $A(0) = 1$, it is enough to check that $A(z)$ satisfies the differential equation $dA/dz = (T + \beta h)A(z)$. Using that T is a derivation, we find

$$TA(z) = \beta \left(\int_0^z dx e^{xT} Th \right) A(z) = \beta ((e^{zT} - 1)h) A(z).$$

On the other hand, $dA/dz = \beta(e^{zT} h)A(z)$, which completes the proof. \square

Combining Lemmas 4.1, 4.3 and 4.4, we get the main result of this section.

Proposition 4.5. *In any vertex algebra or Poisson vertex algebra solving Problem 1.1, with a homogeneous polynomial P of degree $d \geq 1$, one of the following two possibilities holds:*

- (a) $\alpha = 0$ and the polynomial $P(\lambda; h, Th, \dots)$ is independent of λ .
- (b) $\alpha \neq 0$, $\beta = 1/\alpha$, and for some $\delta \in \mathbb{C}$:

$$P(\lambda; h, Th, \dots) = \delta (\lambda + T + \beta h)^d 1. \quad (4.7)$$

Conversely, given (1.15), if either (a) or (b) holds, then the Jacobi identity for the triple (h, e, f) is satisfied. Moreover, if (b) holds, then the Jacobi identity for the triple (f, f, e) follows from the Jacobi identity for the triple (e, e, f) .

Remark 4.6. The above results can be reformulated in terms of the elementary Schur polynomials as follows. Let T be the derivation of the polynomial ring $\mathbb{C}[y_1, y_2, \dots]$ defined by $Ty_k = (k+1)y_{k+1}$. Then one has

$$S_n(y_1, y_2, \dots) = \frac{1}{n!} (y_1 + T)^n 1,$$

which can be derived from the recursive relation

$$S_n(y_1, y_2, \dots) = n(y_1 + T) S_{n-1}(y_1, y_2, \dots).$$

Due to (4.5), the latter is equivalent to the obvious identity

$$(y_1 + T) \exp\left(\sum_{k=1}^{\infty} z^k y_k\right) = \partial_z \exp\left(\sum_{k=1}^{\infty} z^k y_k\right).$$

Equation (4.7) (or (4.6)) gives the most general λ -bracket of e and f that satisfies the Jacobi identity of type hef for $\alpha \neq 0$. By rescaling the generator e , we can assume that the constant $\delta = 1$. Next, we are going to impose the Jacobi identity of type $ee f$. In the vertex algebra case, we will do this separately for $\alpha \neq 0$ and $\alpha = 0$ in Sections 5 and 6, respectively. The Poisson case will be discussed in Section 7.

5. Jacobi identity for e, e, f with $\alpha \neq 0$

In this section, we will work in a vertex algebra solving Problem 1.1, or a non-linear Lie conformal algebra solving Problem 1.2. We will assume that $\alpha \neq 0$, and we let $\beta = 1/\alpha$. The results of the previous section determine the λ -bracket $[e_\lambda f]$, i.e., the polynomial P from (1.15). We are going to impose the Jacobi identity of type $ee f$:

$$[e_\lambda [e_\mu f]] = (-1)^{p(e)} [e_\mu [e_\lambda f]], \quad (5.1)$$

which will determine the possible values of α .

It follows from Proposition 4.5 that equation (5.1) can be deduced from the λ -brackets (1.15) and the axioms of vertex algebra if and only if we have a non-linear Lie conformal algebra $R = \mathbb{C}[T]\langle h, e, f \rangle$ satisfying the assumptions of Problem 1.2. More generally, equation (5.1) may hold under some additional relations among the generators h, e, f and their derivatives; in this case we may have a vertex algebra satisfying the assumptions of Problem 1.1, which is not freely generated. In conclusion, in order to solve Problems 1.1 and 1.2 (namely, in order to prove Theorems 2.2 and 2.3) we need to study equation (5.1).

5.1. Proof of Proposition 2.1

We are given the λ -bracket (1.15) and

$$[e_\lambda f] = :P(\lambda; h, Th, \dots) = :p(\lambda + T - h)1:,$$

where the generators h, e, f are all even, $\alpha = -1$ and $p \in \mathbb{C}[\lambda]$ is an arbitrary polynomial. We need to check the Jacobi identities for the triples (h, e, f) , (e, e, f) , and (f, f, e) .

As before, let $\beta = 1/\alpha = -1$. The Jacobi identity for (h, e, f) follows immediately from Proposition 4.5. Next, applying Corollary 3.3 for the elements $a = e$, $b = -h$, we obtain:

$$[e_\lambda [e_\mu f]] = :p'(\lambda + \mu + T - h)e:,$$

which makes the Jacobi identity of type $ee f$ obvious (see (5.1)). The one of type $f f e$ can be checked in a similar way, or, alternatively, derived from symmetry considerations. Indeed, by the remarks in Section 3.1, we can assume that $p(\lambda) = \lambda^d$ is homogeneous. Then Lemma 4.3 implies $\tilde{P} = (-1)^{d+1}P$, and the Jacobi identity of type $f f e$ follows from the observations in Section 3.1. Therefore, $R = \mathbb{C}[T]\langle h, e, f \rangle$ satisfies all axioms of a non-linear Lie conformal algebra. Recall that when $p(\lambda) = \lambda^d$ this algebra is denoted as R_{-1}^d . \square

5.2. Computation of the commutator

As before, we assume that the polynomial P is homogeneous of degree $d \geq 1$ with respect to the grading given by $\deg \lambda = \deg T = \deg h = 1$. Then P is determined up to a constant by Proposition 4.5(b). By rescaling the generator e , we can take the constant $\delta = 1/d!$ in (4.7) (corresponding to $\gamma = 1$ in (4.6)), so that

$$[e_\lambda f] = :P(\lambda; h, Th, \dots): = \text{Res}_z \frac{e^{z\lambda}}{z^{d+1}} :e^{z(T+\beta h)} 1: . \quad (5.2)$$

To find the double commutator $[e_\lambda[e_\mu f]]$, we apply Proposition 3.2 for the elements $a = e$, $b = \beta h$, and we obtain from equation (3.8) that

$$[e_\lambda :e^{z(T+\beta h)} 1:] = :e^{z(T+\beta h)} 1: (\Psi_\beta(z, \lambda + T)e) :,$$

where the function Ψ_β is defined by (3.5). Therefore,

$$[e_\lambda[e_\mu f]] = \text{Res}_z \frac{e^{z\mu}}{z^{d+1}} :e^{z(T+\beta h)} 1: (\Psi_\beta(z, \lambda + T)e) :. \quad (5.3)$$

Now we will compute the coefficients of certain powers of λ and μ in this formula.

Lemma 5.1. *The right-hand side of (5.3) has the following expansion as a polynomial of λ, μ :*

$$\begin{aligned} & \sum_{k=0}^{d-1} \frac{\lambda^k}{k!} \frac{\mu^{d-1-k}}{(d-1-k)!} \binom{\beta}{k+1} (-1)^{k+1} e \\ & + \frac{\lambda^{d-2}}{(d-2)!} \left\{ \binom{\beta}{d-1} (-1)^{d-1} \beta :he: + \binom{\beta}{d} (-1)^d T e \right\} \\ & + \frac{\mu^{d-2}}{(d-2)!} \left\{ -\beta^2 :he: + \binom{\beta}{2} T e \right\} + \dots, \end{aligned}$$

where the dots denote other powers of λ, μ .

Proof. From (3.5), we obtain the expansion

$$\frac{e^{z\mu}}{z^{d+1}} \Psi_\beta(z, \lambda + T) = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} z^{l+n-d-1} \binom{\beta}{n} (-1)^n \frac{\mu^l}{l!} \frac{(\lambda + T)^{n-1}}{(n-1)!}. \quad (5.4)$$

In order to get the coefficient of $\lambda^k \mu^{d-1-k}$ in (5.4), we need $l = d-1-k$ and $n-1 \geq k$. Then the power of z is $n-k-2 \geq -1$. Therefore, under the residue in z we must have $n-1 = k$ and the exponential $e^{z(T+\beta h)}$ in (5.3) can be replaced by 1. Hence, the coefficient in front of $\frac{\lambda^k}{k!} \frac{\mu^{d-1-k}}{(d-1-k)!}$ in (5.3) is exactly $\binom{\beta}{k+1} (-1)^{k+1} e$.

Similarly, to get the coefficient of λ^{d-2} in (5.4), we need $l = 0$ and $n-1 \geq d-2$. Then the power of z is $n-d-1 \geq -2$. Hence, only the terms with $n = d-1$ and $n = d$ in (5.4) contribute to the residue with respect to z . We thus obtain that the coefficient in front of $\frac{\lambda^{d-2}}{(d-2)!}$ in (5.3) is

$$\binom{\beta}{d-1} (-1)^{d-1} :((T + \beta h)1) e: + \binom{\beta}{d} (-1)^d T e,$$

as claimed.

Finally, in order to have the coefficient of μ^{d-2} in (5.4), we need $l = d - 2$. Then the power of z is $n - 3 \geq -2$. Hence, the only terms in (5.4) contributing to the residue in z are for $n = 1$ and $n = 2$. It follows that the coefficient of $\frac{\mu^{d-2}}{(d-2)!}$ in (5.3) is

$$-\beta((T + \beta h)1)e + \binom{\beta}{2}Te,$$

thus completing the proof. \square

5.3. Proof of Theorems 2.2 and 2.3 for $\alpha \neq 0$

First of all, notice that by Proposition 2.1, R_{-1}^d is indeed a non-linear Lie conformal algebra solving Problem 1.2. Then $V_{-1}^d := V(R_{-1}^d)$ is a vertex algebra solving Problem 1.1. By the results of Section 2.1, the lattice vertex algebra $V_{\mathbb{Z}\sqrt{d+1}}$ also solves Problem 1.1.

Now consider a vertex algebra solving Problem 1.1, or a non-linear Lie conformal algebra solving Problem 1.2, with a homogeneous polynomial P of degree $d \geq 1$. As discussed at the beginning of Section 5.2, the λ -bracket $[e_\lambda f]$ is given by (5.2), and this determines the polynomial P . To find the possible values of α , we will impose the Jacobi identity (5.1), where $[e_\lambda[e_\mu f]]$ is given by (5.3). Let us compare the coefficients of $\frac{\lambda^k}{k!} \frac{\mu^{d-1-k}}{(d-1-k)!}$ in both sides of (5.1) for $k = 0, 1, \dots, d-1$. Then Lemma 5.1 leads to the following system of equations:

$$\binom{\beta}{k+1}(-1)^{k+1} = (-1)^{p(e)} \binom{\beta}{d-k}(-1)^{d-k}, \quad k = 0, 1, \dots, d-1. \quad (5.5)$$

For $d = 1$, we obtain a single equation, $\beta = (-1)^{p(e)}\beta$, which, since $\beta \neq 0$, gives $p(e) = \bar{0}$ and is then satisfied for every β . In this case, by the symmetry (3.2), we also have $p(f) = \bar{0}$. After rescaling, this produces the current Lie conformal algebra or the corresponding vertex algebra (see parts (a) in Theorems 2.2 and 2.3).

The case $d \geq 2$ is treated in the following lemma.

Lemma 5.2. *Given an integer $d \geq 2$ and $p(e) \in \mathbb{Z}/2\mathbb{Z}$, all nonzero complex numbers β satisfying equations (5.5) are:*

- (a) $\beta = -1$, if $p(e) = \bar{0}$.
- (b) $\beta = d + 1$, if $p(e) = \overline{d+1}$.

Proof. It is easy to check that (a) and (b) are indeed solutions of equations (5.5). Now assume that $\beta \neq 0$ is a solution. Our first simple observation is that $\beta \notin \{1, \dots, d-1\}$. For otherwise, taking $k = 0$ in (5.5) would give $\beta = 0$. Next, we will consider separately the cases when d is odd or even.

Assume first that $d = 2n + 1$ is odd. Then (5.5) with $k = n$ implies $p(e) = \bar{0}$. Next, (5.5) with $k = n - 1$ gives $\binom{\beta}{n} = \binom{\beta}{n+2}$, which is equivalent to

$$\frac{(\beta - n)(\beta - n - 1)}{(n + 1)(n + 2)} = 1.$$

The latter has only two solutions: $\beta = -1$ and $\beta = 2n + 2 = d + 1$.

Now if $d = 2n$ is even, equation (5.5) for $k = n$ gives $\binom{\beta}{n+1} = \binom{\beta}{n}(-1)^{p(e)+1}$, which is equivalent to

$$\frac{\beta - n}{n + 1} = (-1)^{p(e)+1}.$$

Therefore, either $p(e) = \bar{0}$ and $\beta = -1$, or $p(e) = \bar{1}$ and $\beta = 2n + 1 = d + 1$. \square

Continuing with the proof of Theorems 2.2 and 2.3, we need to consider the two cases from Lemma 5.2. When $\beta = -1$ and $p(e) = \bar{0}$, we have $\alpha = -1$, and by the symmetry (3.2), $p(f) = \bar{0}$. This gives the λ -brackets of the non-linear Lie conformal algebra R_{-1}^d . The associated enveloping vertex algebra $V_{-1}^d = V(R_{-1}^d)$ is freely generated. By the results of [DSK1], any other vertex algebra satisfying the same λ -bracket relations among the generators is a quotient of V_{-1}^d . In this way, we obtain parts (b) in Theorems 2.2 and 2.3.

When $\beta = d + 1$ and $p(e) = \overline{d+1}$, we compare the coefficients of $\lambda^{(d-2)}$ in both sides of (5.1), using again Lemma 5.1. We obtain the equation

$$\begin{aligned} (d+1) \binom{d+1}{d-1} (-1)^{d-1} :he: + \binom{d+1}{d} (-1)^d Te \\ = (-1)^{d+1} \left(-(d+1)^2 :he: + \binom{d+1}{2} Te \right), \end{aligned}$$

which simplifies to $Te = (d+1) :he:$. This is exactly the first relation in (2.4), and the second one can be deduced by symmetry considerations (see Section 3.1). Because (2.4) are additional relations, not part of the vertex algebra axioms, it follows that non-linear Lie conformal algebras solving Problem 1.2 with $\alpha = (d+1)^{-1}$ do *not* exist. This completes the proof of Theorem 2.2 in the case when $\alpha \neq 0$.

We showed that, in any vertex algebra solving Problem 1.1 with $\alpha = (d+1)^{-1}$, the generators satisfy relations (2.4). In addition, by the results of Section 4.2, the polynomial P is given by (2.2). It is known that there exists a unique (up to isomorphism) such a vertex algebra, namely the lattice vertex algebra $V_{\mathbb{Z}\sqrt{d+1}}$ (see e.g. Sections 5.4 and 5.5 in [K]). This completes the proof of Theorem 2.3.

Remark 5.3. From the above proof we can deduce that, in addition of being not freely generated, the lattice vertex algebra $V_{\mathbb{Z}\sqrt{d+1}}$ cannot be realized as a quotient of a freely generated vertex algebra by an “irregular” ideal (see [DSK1] for the definition).

To complete the proof of Theorem 2.2, we have to consider the case $\alpha = 0$. This will be done in the next section.

6. Jacobi identity for e, e, f with $\alpha = 0$

Now we will consider a non-linear Lie conformal algebra R solving Problem 1.2 with $\alpha = 0$. We denote by $V = V(R)$ its universal enveloping vertex algebra, which is freely generated by the elements h, e, f .

6.1. Differential polynomials in h

Consider the polynomial ring $\mathbb{C}[h, Th, \dots]$, equipped with the derivation T , and the subalgebra $\langle h \rangle \subset V$ generated by h .

Lemma 6.1. *In any non-linear Lie conformal algebra solving Problem 1.2 with $\alpha = 0$, we have:*

- (a) *The λ -bracket of any two elements of $\langle h \rangle$ is zero.*
- (b) *The normally ordered product in $\langle h \rangle$ is commutative and associative.*
- (c) *The map $\mathbb{C}[h, Th, \dots] \rightarrow \langle h \rangle$ defined by (1.17) is an associative algebra isomorphism.*

Proof. First of all, $[h_\lambda h] = 0$ implies $[h_\lambda (T^k h)] = 0$ for all $k \in \mathbb{Z}_+$. From the Wick formula (1.8) we deduce by induction that $[h_\lambda : (T^{k_1} h) \cdots (T^{k_s} h) :] = 0$ for arbitrary $k_1, \dots, k_s \in \mathbb{Z}_+$. Then by the skew-symmetry (1.4), we deduce that $[a_\lambda b] = 0$ for all $a, b \in \langle h \rangle$. This proves part (a). Part (b) follows from (a), the quasi-commutativity (1.6), and quasi-associativity (1.7). Then (c) is immediate from (b) and the fact that $\langle h \rangle$ is freely generated. \square

We introduce the elements

$$H^k = (T + h)^k 1 \in \mathbb{C}[h, Th, T^2 h, \dots], \quad k \in \mathbb{Z}_+,$$

where on the right we use that $T1 = 0$. For example,

$$H^0 = 1, \quad H^1 = h, \quad H^2 = Th + h^2, \quad H^3 = T^2 h + 3h(Th) + h^3.$$

Notice that $H^{k+1} = T^k h +$ terms involving lower order derivatives of h . Thus, we have an isomorphism $\mathbb{C}[h, Th, \dots] \simeq \mathbb{C}[H^1, H^2, \dots]$. As usual, the image of H^k in $\langle h \rangle$ under the map (1.17) will be denoted as $:H^k: = : (T + h)^k 1 :$. In particular, $:H^0: = :1: = |0\rangle$. Similar remarks apply to the elements $\tilde{H}^k = (T - h)^k 1$. Notice that, up to a sign, \tilde{H}^k is the image of H^k under the change of variables (3.2).

Applying Corollary 3.3 for $\beta = 1$ and the elements $a = e$, $b = h$, we obtain:

$$[e_\lambda : H^k :] = -k : H^{k-1} e :, \quad k \in \mathbb{Z}_+. \quad (6.1)$$

Here we take, as usual, normally ordered products of more than two elements from right to left. Similarly, Corollary 3.3 for $\beta = 1$ and $a = f$, $b = -h$ gives

$$[f_\lambda : \tilde{H}^k :] = -k : \tilde{H}^{k-1} f :, \quad k \in \mathbb{Z}_+.$$

This equation can also be deduced from (6.1) by applying the symmetry (3.2).

For any polynomial $v(\lambda) = \sum_{n=0}^N v_n \lambda^n \in V[\lambda]$ with $v_N \neq 0$, we will denote by $\ell.t. v(\lambda) := v_N \lambda^N$ its *leading term*.

Lemma 6.2. *For every $A \in \mathbb{C}[h, Th, \dots]$, $b \in \langle h \rangle$, and $k_1, \dots, k_s \in \mathbb{N}$, we have:*

- (a) $\ell.t. [Ae :_\lambda b] = :A(\ell.t. [e_\lambda b]):,$
- (b) $\ell.t. [e_\lambda : H^{k_1} \cdots H^{k_s} :] = \frac{\lambda^{s-1}}{(s-1)!} (-1)^s k_1 \cdots k_s : H^{k_1-1} \cdots H^{k_s-1} e :.$

Proof. Part (a) is trivial for $A = 1$. By induction, we are going to prove it for monomials $A = (T^{k_1}h) \cdots (T^{k_s}h)$. Write

$$A = (T^k h)C, \quad \text{where } k = k_1, C = (T^{k_2}h) \cdots (T^{k_s}h),$$

and $C = 1$ when $s = 1$. Then $:Ae: = :(T^k h)(:Ce:):$ by the convention of taking normally ordered product from right to left (cf. (1.9)). Using the right Wick formula (1.11) and Lemma 6.1(a), we obtain

$$[:Ae:_\lambda b] = [(T^k h)(:Ce:)_\lambda b] = :(e^{T\partial_\lambda} T^k h)[:Ce:_\lambda b]:.$$

Taking the leading terms and applying the inductive assumption, we get

$$\ell.t. [:Ae:_\lambda b] = :(T^k h)(\ell.t. [:Ce:_\lambda b]): = :(T^k h)(:C(\ell.t. [e_\lambda b]):): = :A(\ell.t. [e_\lambda b]):,$$

which proves part (a).

For $s = 1$, part (b) follows immediately from (6.1). For $s \geq 2$, we proceed by induction on s . Due to Lemma 6.1(b), we can write

$$a = :H^{k_1} \cdots H^{k_s}: = :H^{k_1} b: = :(H^{k_1}): b:, \quad b = :H^{k_2} \cdots H^{k_s}:.$$

Then applying the noncommutative Wick formula (1.8), we obtain

$$[e_\lambda a] = :[e_\lambda :H^{k_1}:] b: + :(H^{k_1}): [e_\lambda b] + \int_0^\lambda d\mu [[e_\lambda :H^{k_1}:]_\mu b].$$

The first two terms in the right-hand side have degree in λ at most $s-2$, by (6.1) and the inductive assumption. Thus,

$$\ell.t. [e_\lambda a] = \ell.t. \int_0^\lambda d\mu [[e_\lambda :H^{k_1}:]_\mu b] = -k_1 \int_0^\lambda d\mu \ell.t. [:H^{k_1-1} e:_\mu b:],$$

where we again used (6.1). Now part (a) and the inductive assumption imply

$$\begin{aligned} \ell.t. [:H^{k_1-1} e:_\mu b]: &= :H^{k_1-1}(\ell.t. [e_\mu b]): \\ &= \frac{\mu^{s-2}}{(s-2)!} (-1)^{s-1} k_2 \cdots k_s :H^{k_1-1} H^{k_2-1} \cdots H^{k_s-1} e:, \end{aligned}$$

which completes the proof. \square

The same proof as above (or the symmetry (3.2)) gives:

$$\ell.t. [f_\lambda : \tilde{H}^{k_1} \cdots \tilde{H}^{k_s}:] = \frac{\lambda^{s-1}}{(s-1)!} (-1)^s k_1 \cdots k_s : \tilde{H}^{k_1-1} \cdots \tilde{H}^{k_s-1} f: . \quad (6.2)$$

6.2. Proof of Theorem 2.2 for $\alpha = 0$

Consider a non-linear Lie conformal algebra solving Problem 1.2 with $\alpha = 0$. By Proposition 4.5, the Jacobi identity for elements h, e, f holds if and only if

$$[e_\lambda f] = :P(h, Th, T^2 h, \dots):$$

is independent of λ . As before, we assume that the polynomial P is homogeneous of degree $d \geq 1$ with respect to the grading given by $\deg T = \deg h = 1$.

Then the Jacobi identity

$$[e_\lambda[e_\mu f]] = (-1)^{p(e)} [e_\mu[e_\lambda f]]$$

implies that the polynomial $[e_\lambda:P(h, Th, \dots)]$ does not depend on λ . Using Lemma 6.2(b), it is easy to see that this is possible only when $P(h, Th, \dots) = \gamma H^d$ for some $\gamma \in \mathbb{C}$. After rescaling the generator e , we can take $\gamma = 1$.

On the other hand, by the skew-symmetry (1.4), we have

$$[f_\lambda e] = -(-1)^{p(e)p(f)} :P(h, Th, \dots):.$$

In the same way as above, using (6.2), the Jacobi identity

$$[f_\lambda[f_\mu e]] = (-1)^{p(f)} [f_\mu[f_\lambda e]]$$

implies that $[f_\lambda e]$ is a scalar multiple of \tilde{H}^d . Therefore, $P(h, Th, \dots) = H^d = \gamma \tilde{H}^d$ for some $\gamma \in \mathbb{C}$, which is possible only for $d = 1$. We thus obtain the current Lie conformal algebra $\mathbb{C}[T]sl_2$ at level 0, thus completing the proof of Theorem 2.2. \square

Remark 6.3. The above proof also works for a vertex algebra V solving Problem 1.1 with $\alpha = 0$, provided the following additional assumptions hold:

- (a) The subalgebra $\langle h \rangle \subset V$ generated by h is freely generated, i.e., the map $\mathbb{C}[h, Th, \dots] \rightarrow V$ defined by (1.17) is injective.
- (b) The similarly defined map $P(h, Th, \dots) \mapsto :P(h, Th, \dots)e:$ is injective.

The conclusion is then that V is a quotient of the universal enveloping vertex algebra of the current Lie conformal algebra $\mathbb{C}[T]sl_2$ at level 0.

7. The Poisson case

Consider a Poisson vertex algebra \mathcal{V} solving Problem 1.1, or a non-linear Poisson conformal algebra R solving Problem 1.2. In the latter case, we denote by $\mathcal{V} = \mathcal{S}(R)$ its universal enveloping Poisson vertex algebra. As before, we assume that the polynomial $P(\lambda; x_1, x_2, \dots)$ from (1.15) is homogeneous of degree $d \geq 1$ with respect to the grading given by $\deg \lambda = 1$, $\deg x_k = k$, where $x_k = T^{k-1}h$.

In this section, we are going to impose the Jacobi identity of type $ee f$ (see (5.1)). Using the Leibniz rule (1.13), we have:

$$\begin{aligned} [e_\lambda[e_\mu f]] &= [e_\lambda P(\mu; h, Th, \dots)] \\ &= \sum_{k=1}^{\infty} [e_\lambda(T^{k-1}h)] \frac{\partial P}{\partial x_k}(\mu; h, Th, \dots) \\ &= - \sum_{k=1}^{\infty} \frac{\partial P}{\partial x_k}(\mu; h, Th, \dots) ((\lambda + T)^{k-1}e). \end{aligned} \tag{7.1}$$

7.1. Proof of Theorem 2.4 for $\alpha = 0$

By Proposition 4.5, for $\alpha = 0$, the Jacobi identity of type hef holds if and only if $P(\lambda; h, Th, \dots) = P(h, Th, \dots)$ is independent of λ . Due to (7.1), equation (5.1) can be written as

$$\sum_{k=1}^{\infty} \frac{\partial P}{\partial x_k}(h, Th, \dots) ((\lambda + T)^{k-1} e) = (-1)^{p(e)} \sum_{k=1}^{\infty} \frac{\partial P}{\partial x_k}(h, Th, \dots) ((\mu + T)^{k-1} e).$$

Since now we assume \mathcal{V} is freely generated, the above equation is equivalent to:

$$p(e) = \bar{0} \quad \text{and} \quad \frac{\partial P}{\partial x_k} = 0, \quad k \geq 2.$$

Therefore, $P(h, Th, \dots) = p(h)$ is a polynomial of h . (When P is homogeneous, after rescaling the generator e , we can assume that $P = h^d$.) Similarly, the Jacobi identity of type fpe gives that $p(f) = \bar{0}$. This proves Theorem 2.4 in the case $\alpha = 0$. \square

Remark 7.1. Let \mathcal{V} be a Poisson vertex algebra solving Problem 1.1 with $\alpha = 0$. Then the above proof works provided the following additional assumptions hold:

- (a) The subalgebra $\langle h \rangle \subset \mathcal{V}$ generated by h is freely generated, i.e., $\langle h \rangle$ is isomorphic to $\mathbb{C}[h, Th, \dots]$.
- (b) The map $\langle h \rangle \rightarrow V$ given by $b \mapsto be$ is injective.

The conclusion is then that \mathcal{V} is a quotient of $\mathcal{S}(R)$, where R is one of the non-linear Poisson conformal algebras described in Theorem 2.4.

7.2. Proof of Theorems 2.4 and 2.5 for $\alpha \neq 0$

Let now $\alpha \neq 0$ and $\beta = 1/\alpha$. By Lemma 4.1, when the polynomial P is homogeneous of degree $d \geq 1$, it is given explicitly by formula (4.3). Then (7.1) can be rewritten as

$$[e\lambda[e_\mu f]] = -\gamma \sum_{k=1}^{\infty} \frac{\partial}{\partial x_k} S_d\left(\mu + \beta \frac{x_1}{1!}, \beta \frac{x_2}{2!}, \beta \frac{x_3}{3!}, \dots\right) ((\lambda + T)^{k-1} e), \quad (7.2)$$

where $x_k = T^{k-1}h$. Using the definition (2.3) of S_d , it is easy to see that the coefficient of λ^{d-1} in the right-hand side of (7.2) is $-\beta\gamma e/d!$, while the coefficient of μ^{d-1} is $-\beta\gamma e/(d-1)!$.

Hence, the Jacobi identity (5.1) implies $d = (-1)^{p(e)}$, which is possible only when $d = 1$ and e is even. By symmetry, f must also be even. In this case, equations (1.15) correspond to the λ -brackets of the current Lie conformal algebra $R = \mathbb{C}[T]sl_2$. This completes the proof of Theorems 2.4 and 2.5. \square

8. Wakimoto realization of V_{-1}^d

In this section, we describe a free-field realization of (a quotient of) the vertex algebra V_{-1}^d defined in Theorem 2.3(b). This gives a representation of V_{-1}^d on the Fock space, which generalizes the Wakimoto realization of the affine vertex algebra of sl_2 at the critical level -2 (see [W]).

8.1. Computations in the Fock space

Consider the non-linear Lie conformal algebra $R = \mathbb{C}[T]\langle a, b \rangle$ with two even generators a, b and λ -brackets

$$[a_\lambda a] = [b_\lambda b] = 0, \quad [a_\lambda b] = -[b_\lambda a] = |0\rangle. \quad (8.1)$$

The corresponding universal enveloping vertex algebra $\mathcal{F} = V(R)$ is called the *Fock space*. The quantum fields $Y(a, z)$ and $Y(b, z)$ corresponding to the elements $a, b \in \mathcal{F}$ are known as *free bosons* (cf. [K, Section 3.5]). By formula (1.2), they satisfy the commutation relations:

$$[Y(a, z), Y(a, w)] = [Y(b, z), Y(b, w)] = 0, \quad [Y(a, z), Y(b, w)] = \delta(z - w).$$

We introduce the following elements of \mathcal{F} :

$$H = -:ab:, \quad E_n = :(H + T)^n a:, \quad F_n = :(H - T)^n b:, \quad n \in \mathbb{Z}_+. \quad (8.2)$$

Lemma 8.1. *For every $m, n \geq 0$, the above elements satisfy:*

$$[H_\lambda a] = -[a_\lambda H] = a, \quad [H_\lambda b] = -[b_\lambda H] = -b, \quad (8.3)$$

$$[H_\lambda H] = -\lambda|0\rangle, \quad [H_\lambda E_n] = E_n, \quad [H_\lambda F_n] = -F_n, \quad (8.4)$$

$$[E_n H] = [F_n H] = 0, \quad \partial_\lambda [E_m H] = \partial_\lambda [F_m H] = 0, \quad (8.5)$$

$$[E_m F_n] = (m + n + 1) : (H - T - \lambda)^{m+n} |0\rangle :, \quad (8.6)$$

$$:E_m F_n: = - : (H - T)^{m+n+1} |0\rangle :. \quad (8.7)$$

Proof. Equations (8.3) follow easily from (8.1), (8.2), the noncommutative Wick formula (1.8), and skew-symmetry (1.4). Indeed, we have, for example,

$$-[a_\lambda H] = [a_\lambda :ab:] = :[a_\lambda a]b: + :a[a_\lambda b]: + \int_0^\lambda d\mu [[a_\lambda a]_\mu b] = a.$$

Similarly,

$$\begin{aligned} -[H_\lambda H] &= [H_\lambda :ab:] = :[H_\lambda a]b: + :a[H_\lambda b]: + \int_0^\lambda d\mu [[H_\lambda a]_\mu b] \\ &=:ab: - :ab: + \lambda|0\rangle = \lambda|0\rangle, \end{aligned}$$

thus proving the first equation in (8.4). The second one, $[H_\lambda E_n] = E_n$, for $n = 0$ becomes $[H_\lambda a] = a$. We will prove it for all $n \geq 0$ by induction. Using that by definition

$$E_{n+1} = :HE_n: + TE_n, \quad F_{n+1} = :HF_n: - TF_n, \quad (8.8)$$

we obtain from the noncommutative Wick formula, sesqui-linearity (1.3), and inductive assumption,

$$[H_\lambda E_{n+1}] = [H_\lambda :HE_n:] + [H_\lambda (TE_n)] = -\lambda E_n + :HE_n: + (\lambda + T)E_n = E_{n+1}.$$

The same argument can be used with F_{n+1} instead of E_{n+1} . This proves (8.4).

Next, we will prove that $\partial_\lambda[E_{m\lambda}E_n] = 0$ by induction on $m + n$. The case $m = n = 0$ is trivial. Assume that $[E_{m\lambda}E_n]$ is independent of λ , namely $[E_{m\lambda}E_n] = E_{m(0)}E_n$. Then, using (8.8) and (8.4), we find

$$\begin{aligned} [E_{m\lambda}E_{n+1}] &= [E_{m\lambda}:HE_n:] + [E_{m\lambda}(TE_n)] \\ &= -:E_mE_n: + :H[E_{m\lambda}E_n]: - \int_0^\lambda d\mu [E_{m\mu}E_n] + (\lambda + T)[E_{m\lambda}E_n] \\ &= -:E_mE_n: + :(H + T)(E_{m(0)}E_n):, \end{aligned}$$

which is constant in λ . By skew-symmetry,

$$[E_{n+1\lambda}E_m] = -[(E_m)_{-\lambda-T}E_{n+1}] = -[E_{m\lambda}E_{n+1}]$$

is also independent of λ . The above equation for $m = n + 1$ implies $[E_{m\lambda}E_m] = 0$, thus completing the proof of (8.5).

We will prove (8.6) and (8.7) simultaneously, again by induction on $m + n$. The case $m = n = 0$ is obvious. Next, we consider the case $m \geq 0, n = 0$. By (8.8) and the quasi-associativity (1.7), we have

$$\begin{aligned} :E_{m+1}F_0: &= :(:HE_m:)F_0: + :(TE_m)F_0: \\ &= :HE_mF_0: + :(\int_0^T d\lambda H)[E_{m\lambda}F_0]: + :(\int_0^T d\lambda E_m)[H_\lambda F_0]: + :(TE_m)F_0:. \end{aligned}$$

The third and fourth term in the right-hand side of this equation cancel each other. By the inductive assumption, the first term is

$$-:H(H - T)^{m+1}|0>: = -:H(H - T)^m H:,$$

while the second term is

$$(m + 1):(\int_0^T d\lambda H)(H - T - \lambda)^m|0>: = -:(H - T)^{m+1}H: + :H(H - T)^{m+1}|0>:,$$

because T is a derivation. We obtain equation (8.7) with $m + 1$ in place of m and $n = 0$. Similarly, using the right Wick formula (1.11) and sesqui-linearity, we find

$$\begin{aligned} [E_{m+1\lambda}F_0] &= [:HE_m:_\lambda F_0] + [(TE_m)_\lambda F_0] \\ &= :(e^{T\partial_\lambda}H)[E_{m\lambda}F_0]: + :(e^{T\partial_\lambda}E_m)[H_\lambda F_0]: + \int_0^\lambda d\mu [E_{m\mu}[H_\lambda F_0]] - \lambda[E_{m\lambda}F_0]. \end{aligned}$$

By the inductive assumption and Taylor's formula, the first term in the right-hand side is

$$(m + 1):(e^{T\partial_\lambda}H)(H - T - \lambda)^m|0>: = (m + 1):(H - T - \lambda)^m H:.$$

It is easy to compute the other three terms and obtain (8.6) with $m + 1$ in place of m and $n = 0$. This proves equations (8.6), (8.7) for $n = 0$.

We are left with proving (8.6) and (8.7) for arbitrary $m, n \geq 0$. Using (8.8), (1.10), and (8.4), we get

$$\begin{aligned} :E_m F_{n+1}: &= :E_m H F_n: - :E_m(T F_n): \\ &= :H E_m F_n: + : \left(\int_{-T}^0 d\lambda [H_\lambda E_m] \right) F_n: - :E_m(T F_n): \\ &= :H E_m F_n: - : (T E_m) F_n: - :E_m(T F_n):. \end{aligned}$$

Since T is a derivation, this equals $:(H - T)(:E_m F_n:)$, as required. Finally, to prove equation (8.6), we compute

$$\begin{aligned} [E_{m\lambda} F_{n+1}] &= [E_{m\lambda} :H F_n:] - [E_{m\lambda}(T F_n)] \\ &= - :E_m F_n: + :H[E_{m\lambda} F_n]: - \int_0^\lambda d\mu [E_{m\mu} F_n] - (\lambda + T)[E_{m\lambda} F_n]. \end{aligned}$$

Using the inductive assumption, it is easy to derive from here (8.6) with $n + 1$ instead of n . This completes the proof of Lemma 8.1. \square

8.2. Wakimoto realization of V_{-1}^d

Recall that the non-linear Lie conformal algebra R_{-1}^d is defined in Theorem 2.2(b), and $V_{-1}^d := V(R_{-1}^d)$ is its universal enveloping vertex algebra.

Theorem 8.2. *For every $d \geq 1$ and every $n = 0, \dots, d$, there is a vertex algebra homomorphism $\pi_n: V_{-1}^d \rightarrow \mathcal{F}$, given by*

$$\pi_n(h) = H, \quad \pi_n(e) = E_n, \quad \pi_n(f) = \frac{(-1)^d}{d+1} F_{d-n}.$$

Proof. Since V_{-1}^d is freely generated, we only need to check that the λ -brackets among the generators h, e, f are preserved. This is done by comparing equations (1.15) with $\alpha = -1$ and (2.6) to equations (8.4)–(8.6) from Lemma 8.1. \square

The map π_n is not injective, because by (8.7) we have

$$:(T - h)^{d+1}|0\rangle: - (d+1):ef: \in \ker \pi_n \quad (8.9)$$

for all $n = 0, \dots, d$. We define *conformal weights* in \mathcal{F} by $\Delta(a) = 1$, $\Delta(b) = 0$, and we obtain from (8.2) and (1.12) that

$$\Delta(H) = 1, \quad \Delta(E_n) = n + 1, \quad \Delta(F_n) = n.$$

Therefore, the map π_n preserves the conformal weight if we let $\Delta(h) = 1$, $\Delta(e) = n + 1$, and $\Delta(f) = d - n$ in V_{-1}^d (cf. (3.1)).

Remark 8.3. The Wakimoto realization [W] of the affine Lie algebra \widehat{sl}_2 at the critical level -2 is given by:

$$\bar{h} = 2H = -2:ab:, \quad e = E_0 = a, \quad f = F_1 = :(H - T)b: = -:ab^2: - 2Tb$$

(in the last equality we used the quasi-associativity (1.7)). Indeed, it follows from Lemma 8.1 that \bar{h}, e, f satisfy equations (1.14) with $k = -2$.

9. Zhu algebra of V_{-1}^d

In this section, we determine the Zhu algebra of the vertex algebra V_{-1}^d with respect to the Hamiltonian operator defined by assigning conformal weights on the generators h, e, f according to (3.1). The result is one of the associative algebras introduced by Smith [S].

9.1. Definition of the Zhu algebra

Recall that a *Hamiltonian operator* H on a vertex algebra V is a diagonalizable linear operator on V such that the *conformal weight* defined by $Ha = \Delta(a)a$ satisfies equations (1.12) for all $j \in \mathbb{Z}$ (see e.g. [K, Section 4.9] for more details). We are also going to use the notation Δ_a for $\Delta(a)$. We introduce the following $*_n$ -products and $*$ -bracket on V (cf. [Z, BK]):

$$a *_n b = \sum_{j \in \mathbb{Z}_+} \binom{\Delta_a}{j} a_{(n+j)} b, \quad n \in \mathbb{Z}, \quad (9.1)$$

$$[a_* b] = \sum_{j \in \mathbb{Z}_+} \binom{\Delta_a - 1}{j} a_{(j)} b = [a_{\partial_x} b] x^{\Delta_a - 1} \Big|_{x=1}. \quad (9.2)$$

The *Zhu algebra* $\text{Zhu}_H V$ of a vertex algebra V with a Hamiltonian operator H is defined as the quotient V/J , where

$$J = \text{span}_{\mathbb{C}} \{a *_{-2} b \mid a, b \in V\}.$$

Then $\text{Zhu}_H V$ is an associative algebra with a product induced by the $*_{-1}$ -product on V (see [Z, Theorem 2.1.1]). This means that in $\text{Zhu}_H V$ we have

$$\pi(a) \pi(b) = \pi(a *_{-1} b), \quad a, b \in V,$$

where π denotes the natural quotient map $V \rightarrow V/J$. Moreover, by [Z, Eq. (2.1.4)], we have

$$\pi(a) \pi(b) - \pi(b) \pi(a) = \pi([a_* b]), \quad a, b \in V.$$

Assume now that $V = V(R)$ is the universal enveloping vertex algebra of a non-linear Lie conformal algebra R , so that the Hamiltonian operator H on V agrees with the grading of R by conformal weight, i.e., $Ha = \Delta_a a$ for $a \in R \subset V$. Then the $*$ -bracket (9.2) is well defined on the quotient $\mathfrak{z} := R/(T + H)R$, and the induced operation $[\ , \]$ on \mathfrak{z} endows it with the structure of a *non-linear Lie algebra* (see [DSK2]). Since V is freely generated by R and

$$a *_{-2} |0\rangle = a_{(-2)} |0\rangle + \Delta_a a_{(-1)} |0\rangle = (T + H)a, \quad a \in V,$$

we can identify \mathfrak{z} as a subspace of $\text{Zhu}_H V$. Then by [DSK2, Corollary 3.26], the Zhu algebra $\text{Zhu}_H V$ is an associative algebra generated by \mathfrak{z} with relations $ab - ba = [a, b]$ for $a, b \in \mathfrak{z}$.

9.2. The Zhu algebra of V_{-1}^d

Recall that $V_{-1}^d = V(R_{-1}^d)$ is the universal enveloping vertex algebra of the non-linear Lie conformal algebra R_{-1}^d defined in Theorem 2.2. Introduce a Hamiltonian operator H on V_{-1}^d such that $\Delta_h = 1$ and $\Delta_e + \Delta_f = d + 1$ (cf. (3.1)). Once we fix the value of Δ_e , this determines H uniquely, because the vertex algebra V_{-1}^d is strongly generated by the elements h, e, f .

For any polynomial $p(\lambda) \in \mathbb{C}[\lambda]$, S.P. Smith investigated in [S] the associative algebra with generators h, e, f and relations

$$he - eh = e, \quad hf - fh = -f, \quad ef - fe = p(h).$$

We will prove that $\text{Zhu}_H V_{-1}^d$ is one of Smith's algebras.

Theorem 9.1. *The Zhu algebra $\text{Zhu}_H V_{-1}^d$ is the associative algebra with generators h, e, f and relations*

$$[h, e] = e, \quad [h, f] = -f, \quad [e, f] = (\Delta_e - h - 1) \cdots (\Delta_e - h - d),$$

where $[a, b] = ab - ba$ denotes commutator with respect to the associative product.

In order to prove Theorem 9.1, we utilize the results of [DSK2] discussed at the end of the previous subsection. Since $R_{-1}^d = \mathbb{C}[T]\langle h, e, f \rangle$, the space $\mathfrak{z} = R/(T + H)R$ is three-dimensional with a basis $\{h, e, f\}$. Here and further, we identify the elements $h, e, f \in R_{-1}^d \subset V_{-1}^d$ with their images under the projection $\pi: V \rightarrow \text{Zhu}_H V_{-1}^d$. By [DSK2, Corollary 3.26], $\text{Zhu}_H V_{-1}^d$ is the associative algebra generated by h, e, f subject to the relations $ab - ba = [a, b] = \pi([a_* b])$. It follows immediately from the definition (9.2) that $[h_* e] = e$ and $[h_* f] = -f$. Therefore, to prove Theorem 9.1, we are left to check that

$$\pi([e_* f]) = d! \binom{\Delta_e - h - 1}{d}. \quad (9.3)$$

We are going to use the following lemma.

Lemma 9.2. *For every $s \geq 1$ and $n_1, \dots, n_s \in \mathbb{Z}_+$, we have*

$$\pi(:(T^{n_1} h) \cdots (T^{n_s} h):) = \pi(T^{n_1} h) \cdots \pi(T^{n_s} h) = (-1)^{n_1 + \cdots + n_s} n_1! \cdots n_s! h^s.$$

Proof. Notice that equations (9.1) and (1.1) imply $a *_{-2} |0\rangle = Ta + \Delta_a a$, and hence $\pi(Ta) = -\Delta_a \pi(a)$. Using this and $\Delta_{Ta} = \Delta_a + 1$, it is easy to check by induction that $\pi(T^n h) = (-1)^n n! h$, which is exactly the statement of the lemma for $s = 1$. We will prove the general case by induction on s . Letting $A = :(T^{n_2} h) \cdots (T^{n_s} h):$, we compute

$$\begin{aligned} (T^n h) *_{-1} A &= \sum_{j \in \mathbb{Z}_+} \binom{\Delta_{T^n h}}{j} (T^n h)_{(j-1)} A \\ &= :(T^n h) A: + \sum_{k=0}^n \binom{n+1}{k+1} (T^n h)_{(k)} A \\ &= :(T^n h) A: + (-1)^n n! h_{(0)} A, \end{aligned}$$

using that by sesqui-linearity $(Ta)_{(k)}b = -k a_{(k-1)}b$. However, since $h_{(0)}h = 0$, we have $h_{(0)}A = 0$. Thus,

$$\pi(:(T^n h)A:) = \pi((T^n h) *_{-1} A) = \pi(T^n h) \pi(A) = (-1)^n n! h \pi(A),$$

completing the proof of the lemma. \square

Proof of Theorem 9.1. As explained above, we only need to check equation (9.3). By the definition (9.2) of the $*$ -bracket, we have

$$\pi([e_* f]) = \pi([e_{\partial_x} f] x^{\Delta_e - 1} \big|_{x=1}) = \pi([e_{\partial_x} f]) x^{\Delta_e - 1} \big|_{x=1},$$

where $[e_{\partial_x} f]$ is obtained by replacing λ with ∂_x in the λ -bracket $[e_\lambda f]$. Recall that, by (2.6) and (4.6) for $\beta = -1$, $\gamma = d!$, we have

$$[e_\lambda f] = :(\lambda + T - h)^d 1: = d! \operatorname{Res}_z \frac{e^{z\lambda}}{z^{d+1}} : \exp\left(-\sum_{k=1}^{\infty} \frac{z^k}{k!} T^{k-1} h\right) :.$$

Applying Lemma 9.2, we obtain

$$\begin{aligned} \frac{1}{d!} \pi([e_\lambda f]) &= \operatorname{Res}_z \frac{e^{z\lambda}}{z^{d+1}} \exp\left(-\sum_{k=1}^{\infty} \frac{z^k}{k!} \pi(T^{k-1} h)\right) \\ &= \operatorname{Res}_z \frac{e^{z\lambda}}{z^{d+1}} \exp\left(\sum_{k=1}^{\infty} \frac{(-z)^k}{k} h\right) = \operatorname{Res}_z \frac{e^{z\lambda}}{z^{d+1}} (1+z)^{-h}, \end{aligned}$$

using the Taylor expansion of $\log(1+z)$. Then

$$\begin{aligned} \frac{1}{d!} \pi([e_* f]) &= \operatorname{Res}_z \frac{e^{z\partial_x}}{z^{d+1}} (1+z)^{-h} x^{\Delta_e - 1} \big|_{x=1} \\ &= \operatorname{Res}_z z^{-d-1} (1+z)^{-h+\Delta_e-1} = \binom{-h+\Delta_e-1}{d}, \end{aligned}$$

which completes the proof of the theorem. \square

9.3. The Zhu algebra of $V_{\mathbb{Z}\sqrt{\beta}}$

The computations of the previous subsection can also be used to determine the Zhu algebra of the lattice vertex algebra $V_{\mathbb{Z}\sqrt{\beta}}$, where $\beta = d+1 \in 2\mathbb{N}$ (see Section 2.1). Here we assume β is even, so that the generators e and f are even, and

$$\Delta_h = 1, \quad \Delta_e = \Delta_f = \beta/2.$$

By Remark 4.2, the λ -bracket $[e_\lambda f]$ in $V_{\mathbb{Z}\sqrt{\beta}}$ is obtained by taking $\beta = d+1$, $\gamma = 1$ in formula (4.6). Then the above proof of Theorem 9.1 gives that $\operatorname{Zhu}_H V_{\mathbb{Z}\sqrt{\beta}}$ is a quotient of the associative algebra with generators h, e, f and relations

$$[h, e] = e, \quad [h, f] = -f, \quad [e, f] = \binom{\beta/2 + \beta h - 1}{\beta - 1}.$$

To obtain $\operatorname{Zhu}_H V_{\mathbb{Z}\sqrt{\beta}}$, we need to quotient by the additional relations (2.4), i.e., by the elements $\pi(Te - \beta :he:)$ and $\pi(Tf + \beta :hf:)$. As in the proof of Lemma 9.2,

we have: $\pi(Te) = -\Delta_e e = -\beta e/2$ and $\pi(Tf) = -\beta f/2$. On the other hand, by (9.1) and (1.15),

$$h *_{-1} e = h_{(-1)}e + h_{(0)}e = :he: + e, \quad h *_{-1} f = :hf: - f.$$

Hence, in $\text{Zhu}_H V_{\mathbb{Z}\sqrt{\beta}}$ we have the relation

$$he = \pi(h *_{-1} e) = \pi(:he:) + e = \frac{1}{\beta} \pi(Te) + e = \frac{e}{2},$$

and similarly $hf = -f/2$. Letting $\beta = 2k$, we obtain precisely the result of [DLM1, Theorem 3.2], since

$$\binom{k + 2kh - 1}{2k - 1} = \frac{2k}{(2k - 1)!} h(4k^2 h^2 - 1)(4k^2 h^2 - 4) \cdots (4k^2 h^2 - (k - 1)^2).$$

Remark 9.3. The above result of [DLM1] shows that the Zhu algebra of the lattice vertex algebra $V_{\mathbb{Z}\sqrt{\beta}}$ is a quotient of a Smith algebra. In [DLM1, Remark 3.3], the authors asked whether one can find a vertex algebra whose Zhu algebra is a Smith algebra and which has $V_{\mathbb{Z}\sqrt{\beta}}$ as a quotient. By Remark 5.3, the answer to that question is negative.

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